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*-QUASIIDEALS IN INVOLUTION SEMIGROUPS

Usama A. Aburawash and **Allam A. Allam**

Department of Mathematics
Faculty of Science-P.O.Box 21511
Alexandria University, Alexandria, Egypt
e-mail:aburawash@alex-sci.edu.eg

Abstract

Let S be an involution semigroup. Then every *-minimal *-quasiideal of S is a minimal *-quasiideal of S. Nevertheless, if S possesses a primitive idempotent e then the *-quasiideal eSe^* (e^*Se) is minimal if and only if it is a (von Neumann) regular *-subsemigroup. Furthermore, S is *-simple if and only if it is simple. Finally, if S has a minimal *-quasiideal then it has a completely simple kernel K and the minimal *-quasiideals of S are just the *-maximal *-subgroups of K.

1. Introduction

Here, we consider only semigroups; that is a nonempty set S with an associative binary operation. An idempotent element f of S is called *primitive* if for every idempotent element $e \in S$, the relation $ef = fe = e$ implies $f = e$. An element $a \in S$ is *regular*, in the sense of von Neumann, if $a \in aSa$.

A semigroup S is said to be *simple* if S is the only ideal of S. A *completely simple semigroup* S is a simple semigroup possessing primitive idempotent. If the intersection K of all ideals of S is nonempty, then K is the unique minimal ideal of S and is called the *kernel* of S (see [5]).

A *quasiideal* Q of S is a nonempty subset of S satisfying $QS \cap SQ \subseteq Q$. Since $Q^2 \subseteq QS \cap SQ \subseteq Q$, it follows that the quasiideal Q is a subsemigroup of S. A quasiideal Q of S is said to be a *minimal quasiideal* if Q does not properly

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contain any quasiideal of S. However, the notion of quasiideal for semigroups was introduced by O. Steinfeld in [4]. For more details we refer to [5].

The following well known results will be essential in proving our results.

Proposition 1.1 *([5], Proposition 2.3). The intersection of a right ideal and a left ideal of a semigroup S is a quasiideal of S.*

Proposition 1.2 *([5], Proposition 2.5). Let e be an idempotent element of a semigroup S. If L (resp. R) is a left (resp. right) ideal of S, then eL (resp. Re) is a quasiideal of S.*

Proposition 1.3 *([5], Proposition 5.3). A quasiideal Q of a semigroup S is minimal if and only if Q is a subgroup of S.*

Proposition 1.4 *([5], Proposition 10.4). Let e be a primitive idempotent element of a semigroup S. If a is a regular element contained in the left (resp. right) ideal Se (resp. eS), then $Sa = Se$ (resp. $aS = eS$).*

2. Semigroups with involution

A semigroup S is said to be an *involution semigroup* or a **-semigroup* if there is defined a unary operation (called *involution*) * on S subject to the identities $a^{**}= a$ and $(ab)^*= b^*a^*$, for all $a, b \in A$. If S is commutative, then it has at least on involution; namely, the identity mapping of S onto S.

A **-quasiideal* (resp. **-ideal*) Q of S will mean a quasiideal (resp. ideal) Q closed under involution; that is $Q^*=\{a^*\in S \mid a \in Q\}\subseteq Q$. A *-quasiideal Q of S is said to be **-minimal if* Q does not properly contain any *-quasiideal of S. Similarly, a *-ideal I of S is **-minimal if* I does not properly contain any *-ideal of S (see[3] or [1]) .

If the intersection K_* of all *-ideals of S is nonempty, then K_* is the unique *-minimal *-ideal of S and is called the **-kernel* of S. It is evident that $K \subseteq K_*$. An involution semigroup S is said to be **-simple* if the only *-ideal of S is S itself.

First, we prove the involutive version of a well known result in semigroups without involution.

Proposition 2.1 *A *-semigroup S is a *-group if and only if S has no proper *-quasiideal.*

Proof If Q is a *-quasiideal of the *-group S, then $S = QS \cap SQ \subseteq Q$ and $S = Q$ follows. Thus S has no proper *-quasiideal. Conversely, assume that S

is a *-semigroup having no proper *-quasiideal. For every $a \in S$, the subsets aS and Sa^* are right and left ideals of S, respectively, whence they are quasiideals of S. By Proposition 1.1, $Q = aS \cap Sa^*$ is a nonempty *-quasiideal of S because $aa^* \in Q$. Thus $Q = S$ and consequently $S \subseteq aS$ which implies $S = aS$. By the same argument, $a^*S \cap Sa$ is a nonempty *-quasiideal of S and again $S = Sa$. Therefore $S = aS = Sa$, which implies that S is a *-group. □

It is evident that a minimal *-quasiideal of an involution semigroup S is *-minimal. The next theorem gives an unexpected result; the converse is also true. First we need the following auxiliary lemma.

Lemma 2.2 *Let Q be a *-minimal *-quasiideal of an involution semigroup S and let $a \in Q$. Then $Q = a^*S \cap Sa$. Moreover, if a is an idemptent element then $a = a^*$ is the identity for Q and $Q = aSa$ is a group.*

Proof For every element $a \in S$, $a^*S \cap Sa$ is a quasiideal of S, by Proposition 1.1, and it is nonempty because it contains a^*a. Moreover, it is a *-quasi ideal of S . Let Q be a *-minimal *-quasiideal of S, and let $a \in Q$, then $a^* \in Q$ and so $a^*S \cap Sa \subseteq Q$. Hence, $a^*S \cap Sa = Q$, by the minimality of Q. If a is an idempotent, a^* is an idempotent, too. By $a^*S \cap Sa = Q$, it follows that a^* and a are left and right identities of Q, respectively and so they coincide and a is the identity of Q. Hence $aSa \subseteq aS \cap Sa = Q$. For any $q \in Q$, $q = ax$ for some $x \in S$ and $q = qa$ because a is the identity of Q, so $q = qa = axa \in aSa$, whence $Q = aSa$. Now we prove that q has a left inverse in Q, namely we know that $q^*S \cap Sq = Q$ and so $a = yq$ for some $y \in Q$. □

Theorem 2.3 *Every *-minimal *-quasiideal Q of an involution semigroup S is a minimal *-quasiideal of S and vice versa.*

Proof Let Q be a *-minimal *-quasiideal of an involution semigroup S and let $a \in Q$, then a^*, $aa^* \in Q$. By Lemma 2.2, $aa^*S \cap Saa^* = Q$, whence $a = aa^*r$ and $a^* = taa^*$ for some $r, t \in S$. Now $ta = taa^*r = a^*r$, whence $ta = tata$ is an idempotent e of Q. So $Q = eQe$ is a subgroup of S, and consequently a minimal quasiideal, according to Proposition 1.3. The converse is obvious. □

Now, we show that if S has a kernel or a *-kernel, then they are coincident.

Proposition 2.4 *If the involution semigroup S has a kernel K or a *-kernel K_*, then $K = K_*$.*

Proof Since $K \cap K^*$ is a nonempty *-ideal of S, then $K \cap K^* = K$ and $K \cap K^* = K_*$, whence $K_* = K$. □

Finally, we give a necessary and sufficient condition for a particular *-quasiideal to be minimal.

Proposition 2.5 *Let S be an involution semigroup and possessing a primitive idempotent e. Then the *-quasiideal eSe^* (e^*Se) is minimal if and only if it is a regular subsemigroup.*

Proof Clearly eSe^* is a nonempty *-quasiideal. Assume first that eSe^* is minimal, then by Proposition 1.3, $eSe^* = G$ is a *-subgroup. Hence for every element $g \in G$, we have $G = gGg$, that is g is regular and so is $G = eSe^*$. For sufficiency, let eSe^* be a regular subsemigroup and Q be a quasiideal of S such that $Q \subseteq eSe^*$. Since e and e^* are idempotents and each element $a \in Q \subseteq eSe^* \subseteq eS \cap Se^*$ is regular, hence, by Proposition 1.4, $eS = aS$ and $Se^* = Sa$. Therefore $eSe^* \subseteq eS \cap Se^* = aS \cap Sa \subseteq Q$, whence $Q = eSe^*$ and consequently eSe^* is minimal. □

3. *-Simple Semigroups with involution

We show first that a *-simple involution semigroup S is simple. However, this result is not true for involution semigroups with 0 (see [2]).

Theorem 3.1 *An involution semigroup S is *-simple if and only if it is simple.*

Proof If S is *-simple and $K \lhd S$ then $K^* \lhd S$ and $K \cap K^*$ is a nonempty *-ideal of S, since $KK^* \subseteq K \cap K^*$. From the*-simplicity of S, it follows that $K \cap K^* = S$. Thus $K = S$ and S is simple. The converse is obvious. □

Finally, we give the involutive version of Theorem 5.14 in [5].

Theorem 3.2 *If the involution semigroup S has a minimal *-quasiideal, then it has a completely simple kernel K. Furthermore, the minimal *-quasiideals of S are just the *-maximal *-subgroups of K.*

Proof The first statement follows immediately from Theorem 5.14 in [5]. Again, by Theorem 5.14 in [5], every minimal *-quasiideal of S is a maximal *-subgroups of K, which is clearly *-maximal. Moreover, a -maximal *-subgroup of K is a minimal *-quasiideal of S, by Proposition 1.3. Nevertheless, if T is not a *-maximal *-subgroup of K, then T is not a maximal *-quasiideal of S, whence by Theorem 5.14 in [5], T is not a minimal *-quasiideal of S. □

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