CLASSIFICATION OF 5-DIMENSIONAL MD-ALGEBRAS HAVING NON-COMMUTATIVE DERIVED IDEALS

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Abstract

The paper presents a subclass of the class of MD5-algebras and MD5-groups, i.e. five dimensional solvable Lie algebras and Lie groups such that their orbits in the co-adjoint representation (K-orbits) are orbits of zero or maximal dimension. The main result of the paper is the classification up to an isomorphism of all MD5-algebras having non-commutative derived ideals.

Introduction

In 1962, studying theory of representations, A. A. Kirillov introduced the Orbit Method (see [2]). This method quickly became the most important method in the theory of representations of Lie groups. Using the Kirillov's Orbit Method, we can obtain all the unitary irreducible representations of solvable and simply connected Lie Groups. The importance of Kirillov's Orbit Method is the co-adjoint representation (K-representation). Therefore, it is meaningful to study the K-representation in the theory of representations of Lie groups.

After studying the Kirillov's Orbit Method, Do Ngoc Diep in 1980 suggested to consider the class of Lie groups and Lie Algebras MD such that the $C^* - algebras$ of them can be described by using KK-functors (see [1]). Let G be an n-dimensional real Lie group. G is called an MDn-group if and

only if its orbits in the K-representation (i.e. K-orbits) are orbits of dimension zero or maximal dimension. The corresponding Lie algebra of G is called an MDn-algebra. Thus, classification and studying of K-representation of the class of MDn-groups and MDn-algebras are the problem of interest. Because all Lie algebras of n dimension (with $n \leq 3$) were listed easily, we have to consider MDn-groups and MDn-algebras with $n \geq 4$. In 1990, all MD4-algebras were classified up to an isomorphism by Vu - the first author (see [5]). Recently, Vu and some his colleagues have continued studying MD5-groups and MD5-algebras having commutative derived ideals (see [6], [7], [8]). In 2008, a classification of all MD5-algebras having commutative derived ideals was given by Vu and Kar Ping Shum (see [9]).

In this paper, we shall give the classification up to an isomorphism of all MD5-algebras $\mathcal{G}$ whose derived ideals $\mathcal{G}^1 := [\mathcal{G}, \mathcal{G}]$ are non-commutative. This classification is the main result of the paper.

The paper is organized as follows: The first section deals with some preliminary notions, section 2 is devoted to the discussion of some results on MDn-algebras, in particular, the main result of the paper is given in this section.

1   Preliminaries

We first recall in this section some preliminary results and notations which will be used later. For details we refer the reader to the book [2] of A. A. Kirillov and the book [1] of Do Ngoc Diep.

1.1   The K-representation and K-orbits

Let G be a Lie group, $\mathcal{G} = \text{Lie}(G)$ be the corresponding Lie algebra of G and $\mathcal{G}^*$ be the dual space of $\mathcal{G}$. For every $g \in G$, we denote the internal automorphism associated with g by $A_{(g)}$, and whence, $A_{(g)} : G \to G$ can be defined as follows $A_{(g)} := g.x.g^{-1}, \forall x \in G$. This automorphism induces the following map $A_{(g)_*} : \mathcal{G} \to \mathcal{G}$ which is defined as follows

$$A_{(g)_*}(X) := \frac{d}{dt} \left[g. \exp\left(tX\right).g^{-1} \right] |_{t=0}, \ \forall X \in \mathcal{G}.$$

This map is called *the tangent map* of $A_{(g)}$. We now formulate the definitions of K-representation and K-orbit.

Definition 1.1.1. *The action*

$$
\begin{aligned}
K : G &\longrightarrow Aut(\mathcal{G}^*) \\
g &\longmapsto K_{(g)}
\end{aligned}
$$

such that

$$\left\langle K_{(g)}(F), X \right\rangle := \left\langle F, A_{(g^{-1})_*}(X) \right\rangle, \forall F \in \mathcal{G}^*, \forall X \in \mathcal{G}$$

is called the co-adjoint representation or K-representation of G in $\mathcal{G}^*$.

Definition 1.1.2. *Each orbit in the co-adjoint representation of G is called a K-orbit of G.*

We denote the K-orbit containing F by Ω_F. Thus, for every $F \in \mathcal{G}^*$, we have $\Omega_F := \{K(g)(F)|g \in G\}$. The dimension of every K-orbit of an arbitrary Lie group G is always even. In order to define the dimension of the K-orbits Ω_F for each F from the dual space $\mathcal{G}^*$ of the Lie algebra $\mathcal{G} = \text{Lie(G)}$, it is useful to consider the following skew-symmetric bilinear form B_F on $\mathcal{G}$: $B_F(X,Y) = \langle F, [X, Y] \rangle, \forall X, Y \in \mathcal{G}$. We denote the stabilizer of F under the co-adjoint representation of G in $\mathcal{G}^*$ by G_F and $\mathcal{G}_F := \text{Lie}(G_F)$.

We shall need in the sequel of the following result.

Proposition 1.1.3 (see [2, Section 15.1]). $KerB_F = \mathcal{G}_F$ *and* $dim\Omega_F = dim\mathcal{G} - dim\mathcal{G}_F = rankB_F$. □

1.2 MD-groups and MD-algebras

Definition 1.2.1 (see [1, Chapter 2]). *An MD-group is a real solvable Lie group of finite dimension such that its K-orbits are orbits of dimension zero or maximal dimension (i.e. dimension k, where k is some even constant and no more than the dimension of the considered group). When the dimension of considered group is n (n is a some positive integer), the group is called an MDn-group. The Lie algebra of an MD-group (MDn-group, respectively) is called an MD-algebra (MDn-algebra, respectively).*

The following proposition gives a necessary condition for a Lie algebra belonging to the class of MD-algebras.

Proposition 1.2.2 (see [3, Theorem 4]). *Let $\mathcal{G}$ be an MD-algebra. Then its second derived ideal $\mathcal{G}^2 := [[\mathcal{G}, \mathcal{G}], [\mathcal{G}, \mathcal{G}]]$ is commutative.* □

We point out here that the converse of the above result is in general not true. In other words, the above necessary condition is not a sufficient condition. So, we now only consider the real solvable Lie algebras having commutative second derived ideals. Thus, they could be MD-algebras.

Proposition 1.2.3 (see [1, Chapter 2, Proposition 2.1]). *Let $\mathcal{G}$ be an MD-algebra with F (in $\mathcal{G}^*$) is not perfectly vanishing in $\mathcal{G}^1 := [\mathcal{G}, \mathcal{G}]$, i.e. there exists $U \in \mathcal{G}^1$ such that $\langle F, U \rangle \neq 0$. Then the K-orbit Ω_F is one of the K-orbits having maximal dimension.* □

2   THE CLASS OF MD5-ALGEBRAS HAVING NON-COMMUTATIVE DERIVED IDEALS

2.1   Some Results on the Class of MD-algebras

In this subsection, we shall present some results on general MDn-algebras ($n \geq 4$).

Firstly, we consider a real solvable Lie algebra $\mathcal{G}$ of dimension n such that $dim\mathcal{G}^1 = n - k$ (k is some integer constant, $1 \leq k < n$), $\mathcal{G}^2$ is non - trivial commutative and $dim\mathcal{G}^2 = dim\mathcal{G}^1 - 1 = n - k - 1$. Without loss of generality, we may assume that

$$\begin{aligned}
\mathcal{G} &= gen\left(X_1, X_2, ..., X_n\right), \ (n \geq 4), \\
\mathcal{G}^1 &= gen\left(X_{k+1}, X_{k+2}, ..., X_n\right), \ (n > k \geq 1), \\
\mathcal{G}^2 &= gen\left(X_{k+2}, ..., X_n\right),
\end{aligned}$$

with the Lie brackets are given by

$$[X_i, X_j] = \sum_{l=k+1}^{n} C_{ij}^l X_l, \ 1 \leq i < j \leq n,$$

where $C_{ij}^l \ (1 \leq i < j \leq n, k+1 \leq l \leq n)$ are constructional constants of $\mathcal{G}$.

Theorem 2.1.1. *There is no MD-algebra $\mathcal{G}$ such that its second derived ideal $\mathcal{G}^2$ is not trivial and less than its first derived ideal $\mathcal{G}^1$ by one dimension: $dim\mathcal{G}^2 = dim\mathcal{G}^1 - 1$.*

In order to prove this theorem, we need some lemmas.

Lemma 2.1.2. *The operator $ad_{X_{k+1}}$ restricted on $\mathcal{G}^2$ is an automorphism.*

Proof. Since $\mathcal{G}^2$ is commutative, $[X_i, X_j] = 0, \ \forall i, j \geq k + 2$. Hence,

$$\begin{aligned}
gen\left(X_{k+2}, X_{k+3}, \cdots, X_n\right) &= \mathcal{G}^2 = \left[\mathcal{G}^1, \mathcal{G}^1\right] \\
&= gen\left(\left[X_{k+1}, X_{k+2}\right], \left[X_{k+1}, X_{k+3}\right], \cdots, \left[X_{k+1}, X_n\right]\right) \\
&= gen\left(ad_{X_{k+1}}\left(X_{k+2}\right), \cdots, ad_{X_{k+1}}\left(X_n\right)\right).
\end{aligned}$$

It follows that $ad_{X_{k+1}}$ restricted on $\mathcal{G}^2$ is automorphic.   □

Lemma 2.1.3. *Without any restriction of generality, we can always suppose right from the start that $[X_i, X_{k+1}] = 0$ for all indices i such that $1 \leq i \leq k$.*

Proof. Firstly, we remark that $[X_1, X_{k+1}] \in \mathcal{G}^1$, so there exists $X \in \mathcal{G}^2$ such that $[X_1, X_{k+1}] = C_{1,k+1}^{k+1} X_{k+1} + X$. Since $ad_{X_{k+1}}$ restricted on $\mathcal{G}^2$ is automorphic, there exists $Y \in \mathcal{G}^2$ such that $ad_{X_{k+1}}(Y) = X$.

By changing $X_1' = X_1 + Y$, we get $[X_1', X_{k+1}] = C_{1,k+1}^{k+1} X_{k+1}$. Using the Jacobi identity for X_1', X_{k+1}, and an arbitrary element $Z \in \mathcal{G}^2$ we obtain $ad_{X_1'} \circ ad_{X_{k+1}} - ad_{X_{k+1}} \circ ad_{X_1'} = \alpha ad_{X_{k+1}}$, where α is some real constant. Since $ad_{X_{k+1}}$ is automorphic on $\mathcal{G}^2$, α must be zero. Therefore, $C_{1,k+1}^{k+1} = 0$, i.e. $[X_1', X_{k+1}] = 0$. So, we can suppose that $[X_1, X_{k+1}] = 0$.

By the same way, we can suppose $[X_2, X_{k+1}] = \cdots = [X_k, X_{k+1}] = 0$.   □

Lemma 2.1.4. $[X_i, X_j] = C_{ij}^{k+1} X_{k+1}$ *for all pairs of indices i, j such that* $1 \leq i < j \leq k$.

Proof. Consider an arbitrary pair of indices i, j such that $1 \leq i < j \leq k$. Note that $[X_i, X_j] = \sum\limits_{l=k+1}^{n} C_{ij}^l X_l = C_{ij}^{k+1} X_{k+1} + \sum\limits_{l=k+2}^{n} C_{ij}^l X_l$. By using the Jacobi identity, we have

$$[X_i, [X_j, X_{k+1}]] + [X_j, [X_{k+1}, X_i]] + [X_{k+1}, [X_i, X_j]] = 0$$

$$\Rightarrow [X_{k+1}, [X_i, X_j]] = \left[X_{k+1}, C_{ij}^{k+1} X_{k+1} + \sum\limits_{l=k+2}^{n} C_{ij}^l X_l \right] = 0$$

$$\Rightarrow ad_{X_{k+1}} \left(\sum\limits_{l=k+2}^{n} C_{ij}^l X_l \right) = 0$$

$$\Rightarrow \sum\limits_{l=k+2}^{n} C_{ij}^l X_l = 0 \text{ (because } ad_{X_{k+1}} \text{ is automorphic on } \mathcal{G}^2)$$

$$\Rightarrow [X_i, X_j] = C_{ij}^{k+1} X_{k+1}; 1 \leq i < j \leq k.$$

□

We now prove Theorem 2.1.1. Namely, we will prove that if $\mathcal{G}$ is a real solvable Lie algebra such that $\mathcal{G}^2$ is non - trivial commutative and $dim\mathcal{G}^2 = dim\mathcal{G}^1 - 1 = n - k - 1$, $1 \leq k < n$, then $\mathcal{G}$ is not an MD-algebra.

Proof of Theorem 2.1.1.

According to above lemmas, we can choose a suitable basis $(X_1, X_2, \cdots, X_n)$ of $\mathcal{G}$ which satisfies the following conditions:

$$\begin{aligned}
[X_i, X_j] \quad &= C_{ij}^{k+1} X_{k+1}, \ 1 \leq i < j \leq k; \\
[X_i, X_{k+1}] \quad &= 0, \ 1 \leq i \leq k; \\
[X_i, X_j] \quad &= \sum\limits_{l=k+2}^{n} C_{ij}^l X_l, \ 1 \leq i \leq k+1, \ k+2 \leq j \leq n.
\end{aligned}$$

Moreover, the constructional constants C_{ij}^{k+1} can not concomitantly vanish and the matrix $A = \left(C_{j,k+1}^l \right)_{k+2 \leq j,l \leq n}$ is invertible because $ad_{X_{k+1}}$ restricted on $\mathcal{G}^2$ is automorphic.

Since A is invertible, there exist $\alpha_{k+2}, \cdots, \alpha_n \in \mathbb{R}$, which are not concomitantly vanishing, such that

$$A \begin{bmatrix} \alpha_{k+2} \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{n-k-1}.$$

Let $(X_1^*, X_2^*, \cdots, X_n^*)$ is the dual basis in $\mathcal{G}^*$ of $(X_1, X_2, \cdots, X_n)$. We choose $F_1 = X_{k+1}^*$ and $F_2 = X_{k+1}^* + \alpha_{k+2}X_{k+2}^* + \cdots + \alpha_n X_n^*$ in $\mathcal{G}^*$. It can easily be seen that F_1, F_2 are not perfectly vanishing in $\mathcal{G}^1$. In view of Proposition 1.2.3, if $\mathcal{G}$ is an MD-algebra then $\Omega_{F_1}, \Omega_{F_2}$ are orbits of maximal dimension, in particular we have

$$rankB_{F_2} = dim\Omega_{F_1} = dim\Omega_{F_2} = rankB_{F_2}.$$

But it is easy to verify that $rankB_{F_2} \geq rankB_{F_1} + 2$. This contradiction proves that $\mathcal{G}$ is not an MD-algebra and the proof of Theorem 2.1.1 is therefore complete. □

Now we consider an arbitrary real solvable Lie algebra $\mathcal{G}$ of dimension n ($n \geq 5$) such that $dim\mathcal{G}^1 = n - 1$. It is obvious that we can choose one basis $(X_1, X_2, \cdots, X_n)$ of $\mathcal{G}$ such that $\mathcal{G}^1 = gen\,(X_2, X_3, \cdots, X_n)$, $\mathcal{G}^2 \subset gen\,(X_3, \cdots, X_n)$ and $\mathcal{G}^2$ is commutative. Let $C_{ij}^l\,(1 \leq i < j \leq n, 2 \leq l \leq n)$ be constructional constants of $\mathcal{G}$. Then the Lie brackets are given by the following formulas

$$[X_i, X_j] = \sum_{l=2}^{n} C_{ij}^l X_l \, (1 \leq i < j \leq n).$$

Theorem 2.1.5. *Let $\mathcal{G}$ be a real solvable Lie algebra of dimension n such that its first derived ideal $\mathcal{G}^1$ is $(n-1)$-dimensional ($n \geq 5$) and its second derived ideal $\mathcal{G}^2$ is commutative. Then $\mathcal{G}$ is MDn-algebra if and only if $\mathcal{G}^1$ is commutative.*

In order to prove this theorem, once again, we also need some lemmas.

Lemma 2.1.6. *If $\mathcal{G}$ is an MD-algebra of dimension n ($n \geq 5$) such that $\dim\mathcal{G}^1 = n - 1$ then $\dim\Omega_F \in \{0, 2\}$ for every $F \in \mathcal{G}^*$.*

Proof. Let $ad_{X_1} = (a_{ij})_{n-1} \in End\,(\mathcal{G}^1)$. With $F_0 = X_2^* \in \mathcal{G}^*$, the matrix of

the bilinear form B_{F_0} in the chosen basis is given as follows

$$B_{F_0} = \begin{bmatrix} 0 & -a_{12} & -a_{13} & \ldots & -a_{1n} \\ a_{12} & 0 & 0 & \ldots & 0 \\ a_{13} & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \\ a_{1n} & 0 & 0 & \ldots & 0 \end{bmatrix}.$$

It is plain that $rank\, B_{F_0} = 2$. Since $\mathcal{G}$ is an MD-algebra, we get $\dim \Omega_F \in \{0, 2\}$ for every $F \in \mathcal{G}^*$. $\qquad\square$

Lemma 2.1.7. *Suppose that $\mathcal{G} = gen\,(X_1, X_2, \cdots, X_n)$ is a real solvable Lie algebra of dimension n such that*
$$\mathcal{G}^1 = gen\,(X_2, X_3, \cdots, X_n) \quad \text{and} \quad \mathcal{G}^2 = gen\,(X_{k+1}, \cdots, X_n),\ \ k > 1.$$
Let
$$A = \begin{pmatrix} C_{12}^2 & \cdots & C_{1k}^2 \\ \vdots & \ddots & \vdots \\ C_{12}^k & \cdots & C_{1k}^k \end{pmatrix}$$

be the matrix established by the constructional constants $C_{1j}^l\ (2 \le j, l \le k)$ of $\mathcal{G}$. Then A is invertible.

Proof. Since $\mathcal{G}^1 = [\mathcal{G}, \mathcal{G}]$, there exist real numbers $\alpha_{ij},\ 1 \le i < j \le n$, such that

$$
\begin{aligned}
X_2 \ &= \ \sum_{1 \le i < j \le n} \alpha_{ij}\,[X_i, X_j] \\
&= \ \sum_{j=2}^{k} \alpha_{1j}\,[X_1, X_j] + \sum_{j=k+1}^{n} \alpha_{1j}\,[X_1, X_j] + \sum_{2 \le i < j \le n} \alpha_{ij}\,[X_i, X_j] \\
&= \ \sum_{j=2}^{k} \alpha_{1j}\,[X_1, X_j] + LC_1\left(\mathcal{G}^2\right) \\
&= \ \sum_{j=2}^{k} \alpha_{1j}\left(\sum_{l=2}^{n} C_{1j}^l X_l\right) + LC_1\left(\mathcal{G}^2\right) \\
&= \ \sum_{j=2}^{k} \alpha_{1j}\left(\sum_{l=2}^{k} C_{1j}^l X_l + \sum_{l=k+1}^{n} C_{1j}^l X_l\right) + LC_1\left(\mathcal{G}^2\right) \\
&= \ \sum_{l=2}^{k}\sum_{j=2}^{k} C_{1j}^l \alpha_{1j} X_l + LC_2\left(\mathcal{G}^2\right),
\end{aligned}
$$

where $LC_1\left(\mathcal{G}^2\right)$, $LC_2\left(\mathcal{G}^2\right)$ are linear combinations of some definite vectors from the chosen basis of $\mathcal{G}^2$. This implies that there exists columnar vector $Y_2 \in \mathbb{R}^{k-1}$ such that

$$AY_2 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{k-1}.$$

Similarly, there exist columnar vectors $Y_3, \cdots, Y_k \in \mathbb{R}^{k-1}$ such that

$$AY_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, AY_k = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{k-1}.$$

Thus, there is a real matrix P such that $A.P = I$, where I is the identity $(k-1)$-matrix. So A is invertible and Lemma 2.1.7 is proved completely. □

Proof of Theorem 2.1.5

Firstly, we shall prove that, if $\mathcal{G} = gen\,(X_1, X_2, \cdots, X_n)\,(n \geq 5)$ such that $\mathcal{G}^1 = gen(X_2, X_3, \cdots, X_n)$ is non-commutative, then $\mathcal{G}$ is not an MD-algebra.

Let $\mathcal{G}^2 = gen\,(X_{k+1}, \cdots, X_n)$, $2 \leq k < n$. We need to consider some cases which contradict each other as follows.

1. $k = 2$. Then, $\dim \mathcal{G}^2 = \dim \mathcal{G}^1 - 1$. According to Theorem 2.1.1, G is not an MD-algebra.

2. $k = 3$. That means that $\mathcal{G}^2 = gen\,(X_4, \cdots, X_n)$. Assume that $\mathcal{G}$ is an MD-algebra. Remember that $[X_1, X_2] = \sum_{l=2}^{n} C_{12}^l X_l$, $[X_1, X_3] = \sum_{l=2}^{n} C_{13}^l X_l$, $[X_i, X_j] = \sum_{l=4}^{n} C_{ij}^l X_l$, for all $j > 4$ when $i = 1$, $j \geq 3$ when $i = 2$ and $j > 3$ when $i = 3$. According to Lemma 2.1.7, the matrix $P = \begin{bmatrix} C_{12}^2 & C_{12}^3 \\ C_{13}^2 & C_{13}^3 \end{bmatrix}$ is invertible.

   Let $F = \alpha_1 X_1^* + \alpha_2 X_2^* + \cdots + \alpha_n X_n^*$ be an arbitrary element of $\mathcal{G}^*$, where $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{R}$. The matrix of the bilinear form B_F is

$$B_F = \begin{bmatrix} 0 & -F\,([X_1, X_2]) & \cdots & -F\,([X_1, X_n]) \\ F\,([X_1, X_2]) & 0 & \cdots & -F\,([X_2, X_n]) \\ \cdots & \cdots & \cdots & \cdots \\ F\,([X_1, X_n]) & F\,([X_2, X_n]) & \cdots & 0 \end{bmatrix}.$$

   Now we consider the 4-submatrices of B_F established by the elements which are on the rows and the columns of the same numbers 1, 2, 3, i $(i > 3)$. Because $\mathcal{G}$ is an MD-algebra, so according to Lemma 2.1.6, we get $rank(B_F) \in \{0, 2\}$, this implies that the determinants of these 4-submatrices are zero for any $F \in \mathcal{G}^*$. By direct computations, using the following obvious result of Linear Algebra: *the determinant of any skew-symmetric real 4-matrix $(a_{ij})_4$ is equal to zero if and only if $a_{12}.a_{34} - $*

$a_{13}.a_{24}+a_{14}.a_{23} = 0$, we get $C_{2i}^l = C_{3i}^l = 0, l \geq 4$. This implies $[X_2, X_i] = [X_3, X_i] = 0, i \geq 4$. Note that $\mathcal{G}^2$ is commutative. So we have

$$\begin{aligned} \mathcal{G}^2 \ &= \left[\mathcal{G}^1, \mathcal{G}^1\right] \\ &= gen\,(X_4, \cdots, X_n) \\ &= gen\,([X_i, X_j]\,; i, j \geq 2) \\ &= gen\,([X_2, X_3])\,. \end{aligned}$$

Thus, $n - 3 = \dim \mathcal{G}^2 \leq 1$, i.e. $n \leq 4$. This contradicts the hypothesis $n \geq 5$. That means $\mathcal{G}$ is not an MD-algebra.

3. $k \geq 4$. By an argument analogous to that used above, we also prove that $\mathcal{G}$ is not an MD-algebra.

Conversely, assume that $\mathcal{G}$ is a real solvable Lie algebra of dimension n such that its first derived ideal is $(n-1)$-dimensional and commutative, i.e. $\mathcal{G}^1 \equiv \mathbb{R}.X_2 \oplus \mathbb{R}.X_3 \oplus ... \oplus \mathbb{R}.X_n \equiv \mathbb{R}^{n-1}$. We need to show that $\mathcal{G}$ is an MD-algebra.

Let $F = \alpha_1 X_1^* + \alpha_2 X_2^* + \cdots + \alpha_n X_n^* \equiv (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{R}^n$ be an arbitrary element from $\mathcal{G}^* \equiv \mathbb{R}^n$, where $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{R}$. By simple computation, we can see that the matrix of the bilinear form B_F is

$$B_F = \begin{bmatrix} 0 & -F\left([X_1, X_2]\right) & \cdots & -F\left([X_1, X_n]\right) \\ F\left([X_1, X_2]\right) & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ F\left([X_1, X_n]\right) & 0 & \cdots & 0 \end{bmatrix}.$$

It is clear that $rankB_F \in \{0, 2\}$. Hence, $\mathcal{G}$ is an MDn-algebra and Theorem 2.1.5 is proved completely. $\qquad\square$

2.2 Classification of MD5-algebras having non-commutative derived ideals

The following theorem is the main result of the paper. It gives the classification up to an isomorphism of MD5-algebras having non-commutative derived ideals.

Theorem 2.2.1. *Let $\mathcal{G}$ be an MD5-algebra such that the first derived ideal $\mathcal{G}^1 = [\mathcal{G}, \mathcal{G}]$ is non-commutative. Then the following assertions hold.*

(i) *If $\mathcal{G}$ is decomposable, then $\mathcal{G} \cong \mathcal{H} \oplus \mathcal{K}$, where $\mathcal{H}$ and $\mathcal{K}$ are MD-algebras of dimensions which are no more than 4.*

(ii) *If $\mathcal{G}$ is indecomposable, then we can choose a suitable basis $(X_1, X_2, X_3, X_4, X_5)$ of $\mathcal{G}$ such that $\mathcal{G}^1 = gen(X_3, X_4, X_5)$, $[X_3, X_4] = X_5$; operators ad_{X_1}, ad_{X_2} act on $\mathcal{G}^1$ as the following endomorphisms*

$$ad_{X_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \ ad_{X_2} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the other Lie brackets are trivial.

We need to prove some lemmas before we prove Theorem 2.2.1.

Lemma 2.2.2. *Let $\mathcal{G}$ be a real solvable Lie algebra. For any $Z \in \mathcal{G}$ we consider ad_Z as an operator acting on $\mathcal{G}^1$. Then we have $Trace\,(ad_Z) = 0$ for all $Z \in \mathcal{G}^1$.*

Proof. Using the Jacobi identity for $X, Y \in \mathcal{G}$ and an arbitrary element $Z \in \mathcal{G}^1$, we have $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$. So, $ad_X \circ ad_Y - ad_Y \circ ad_X = ad_{[X,Y]}$. This implies $Trace\,(ad_{[X,Y]}) = 0$. Note that $\mathcal{G}^1 = [\mathcal{G}, \mathcal{G}]$ and ad_Z is a linear map. So we get $Trace\,(ad_Z) = 0$ for all $Z \in \mathcal{G}^1$. $\qquad\square$

Lemma 2.2.3. *If $\mathcal{G}$ is a real solvable Lie algebra with $\dim \mathcal{G}^1 = 2$ then $\mathcal{G}^1$ is commutative.*

Proof. We choose a basis (X, Y) of $\mathcal{G}^1$. Assume that $[X, Y] = aX + bY$. So we have $ad_X = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}$, $ad_Y = \begin{pmatrix} -a & 0 \\ -b & 0 \end{pmatrix} \in End(\mathcal{G}^1)$. According to Lemma 2.2.2, we get $a = b = 0$. Hence, $\mathcal{G}^1$ is commutative. $\qquad\square$

Now we are ready to prove Theorem 2.2.1 - The main result of the paper.

Proof of Theorem 2..2.1

It is clear that assertion (i) of Theorem 2.2.1 holds obviously. We only need to prove assertion (ii). Let $\mathcal{G}$ be an indecomposable MD5-algebra such that the first derived ideal $\mathcal{G}^1 = [\mathcal{G}, \mathcal{G}]$ is non - commutative and the second derived ideal $\mathcal{G}^2 = [\mathcal{G}^1, \mathcal{G}^1]$ is commutative. According to Theorems 2.1.1, 2.1.5 and Lemma 2.2.3, the dimensions of $\mathcal{G}^1$ and $\mathcal{G}^2 = [\mathcal{G}^1, \mathcal{G}^1]$ must be 3 and 1, respectively. We choose a basis $(X_1, X_2, X_3, X_4, X_5)$ such that $\mathcal{G}^1 = gen\,(X_3, X_4, X_5)$ and $\mathcal{G}^2 = gen\,(X_5)$ with the Lie brackets are given by

$$
\begin{aligned}
&[X_1, X_2] = a_3 X_3 + a_4 X_4 + a_5 X_5, \\
&[X_1, X_3] = b_3 X_3 + b_4 X_4 + b_5 X_5, \\
&[X_1, X_4] = c_3 X_3 + c_4 X_4 + c_5 X_5, \\
&[X_1, X_5] = d_3 X_3 + d_4 X_4 + d_5 X_5, \\
&[X_2, X_3] = e_3 X_3 + e_4 X_4 + e_5 X_5, \\
&[X_2, X_4] = f_3 X_3 + f_4 X_4 + f_5 X_5, \\
&[X_2, X_5] = k_3 X_3 + k_4 X_4 + k_5 X_5, \\
&[X_3, X_4] = g_5 X_5, [X_3, X_5] = h_5 X_5, [X_4, X_5] = l_5 X_5,
\end{aligned}
$$

where $a_i, b_i, c_i, d_i, e_i, f_i, k_i$ $(i = 3, 4, 5)$ and g_5, h_5, l_5 are the definite real numbers.

Now we give some useful remarks as follows.

   a. According to Lemma 2.2.2, $Trace\,(ad_{X_3}) = Trace\,(ad_{X_4}) = 0$. That means $h_5 = l_5 = 0$.

b. $g_5 \neq 0$ because $\mathcal{G}^2 = gen(X_5)$. By changing X_3 with $X_3' = \frac{1}{g_5}X_3$, we get $[X_3', X_4] = X_5$. So, we can suppose right from the start that $[X_3, X_4] = X_5$, i.e. $g_5 = 1$.

c. $d_3 = d_4 = k_3 = k_4 = 0$ because $\mathcal{G}^2 = \mathbb{R}.X_5$ is an ideal of $\mathcal{G}$. So, we get

$$[X_1, X_5] = d_5 X_5, \quad [X_2, X_5] = k_5 X_5.$$

If $k_5 \neq 0$, by changing $X_2' = X_1 - \frac{d_5}{k_5}X_2$ we get $[X_2', X_5] = 0$. So, we can always assume that $k_5 = 0$.

d. By changing X_1 with $X_1' = X_1 - c_5 X_3 + b_5 X_4$ and X_2 with $X_2' = X_2 - f_5 X_3 + e_5 X_4$, we get
$$[X_1', X_3] = b_3 X_3 + b_4 X_4, \quad [X_1', X_4] = c_3 X_3 + c_4 X_4,$$
$$[X_2', X_3] = e_3 X_3 + e_4 X_4, \quad [X_2', X_4] = f_3 X_3 + f_4 X_4.$$

Thus, we can suppose right from the start that $b_5 = c_5 = e_5 = f_5 = 0$.

Using the Jacobi identity for triads X_1, X_2, X_i $(i = 3, 4, 5)$, we obtain

$$(I) \begin{cases} a_3 = a_4 = 0, \\ e_4 c_3 = b_4 f_3, \\ e_3 b_4 + e_4 c_4 = b_3 e_4 + b_4 f_4, \\ f_3 b_3 + f_4 c_3 = c_3 e_3 + c_4 f_3, \\ b_3 + c_4 = d_5, \\ e_3 + f_4 = 0. \end{cases}$$

So we can reduce the Lie brackets as follows

$$\begin{aligned}
[X_1; X_2] &= & & & a_5 X_5, \\
[X_1; X_3] &= & b_3 X_3 &+& b_4 X_4, \\
[X_1; X_4] &= & c_3 X_3 &+& c_4 X_4, \\
[X_1; X_5] &= & & & (b_3 + c_4) X_5, \\
[X_2; X_3] &= & e_3 X_3 &+& e_4 X_4, \\
[X_2; X_4] &= & f_3 X_3 &-& e_3 X_4, \\
[X_3; X_4] &= & & & X_5.
\end{aligned}$$

Thus, Relations (I) can be rewritten as follows

$$(II) \begin{cases} e_4 c_3 = b_4 f_3, \\ 2e_3 b_4 = e_4 (b_3 - c_4), \\ 2c_3 e_3 = f_3 (b_3 - c_4). \end{cases}$$

Now we need to consider the following cases which contradict each other.

Case 1: $e_3 = e_4 = 0$.

$$(II) \Leftrightarrow \begin{cases} b_4 f_3 = 0, \\ f_3 (b_3 - c_4) = 0. \end{cases}$$

1.1. Assume that $f_3 = 0$. Then, Relations (II) is automatically satisfied.

By choosing $F_1 = X_3^* \in \mathcal{G}^*$, we get $rankB_{F_1} = 2$.

Now we choose $F_2 = X_5^* \in \mathcal{G}^*$. By simple computation, we obtain

$$B_{F_2} = \begin{bmatrix} 0 & -a_5 & 0 & 0 & -(b_3 + c_4) \\ a_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ b_3 + c_4 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since $\mathcal{G}$ is an MD-algebra, this implies that $rankB_{F_2} = rankB_{F_1} = 2$. That fact implies that $a_5 = b_3 + c_4 = 0$, so $\mathcal{G}$ is decomposable, which is a contradiction. Thus, this case cannot happen.

1.2. Now we assume that $f_3 \neq 0$. Then $b_4 = 0, b_3 = c_4$.
By changing X_1 and X_2 with $X_1' = X_1 - c_3 X_2'$ and $X_2' = \frac{1}{f_3} X_2$ we can suppose right from the start that $f_3 = 1, c_3 = 0$. Because the dimension of $\mathcal{G}^1$ is 3, $b_3 \neq 0$. By changing X_1 with $X_1' = \frac{1}{b_3} X_1$, we can always assume that $b_3 = 1$. By changing X_2 with $X_2' = X_2 - \frac{a_5}{2} X_5$, we can assume that $a_5 = 0$. Now, we choose $F_3 = X_3^*$ and $F_4 = X_4^*$ from $\mathcal{G}^*$, we get $rankB_{F_3} = 2, rankB_{F_4} = 4$. This cannot happen because $\mathcal{G}$ is an MD-algebra.

Case 2: $e_4 = 0, e_3 \neq 0$

$$(II) \Leftrightarrow \begin{cases} b_4 f_3 = 0, \\ b_4 e_3 = 0, \\ 2c_3 e_3 = f_3 (b_3 - c_4). \end{cases} \Leftrightarrow \begin{cases} b_4 = 0, \\ 2c_3 e_3 = f_3 (b_3 - c_4). \end{cases}$$

By changing X_2 with $X_2' = \frac{1}{e_3} X_2$, we can assume that $e_3 = 1$. Now Relations (II) can be rewritten as follows

$$(III) \begin{cases} b_4 = 0, \\ 2c_3 = f_3 (b_3 - c_4). \end{cases}$$

By changing X_4 with $X_4' = X_4 - \frac{f_3}{2} X_3$ and X_1 with $X_1' = X_1 - b_3 X_2$, we can suppose that $b_3 = f_3 = 0$. From Relations (III) we get $b_4 = c_3 = 0$. Let $F = \alpha X_1^* + \beta X_2^* + \gamma X_3^* + \delta X_4^* + \sigma X_5^* \equiv (\alpha, \beta, \gamma, \delta, \sigma)$ be an arbitrary element from $\mathcal{G}^* \equiv \mathbb{R}^5$; $\alpha, \beta, \gamma, \delta, \sigma \in \mathbb{R}$. By simple computation, we

obtain the matrix of the bilinear form B_F as follows

$$B_F = \begin{bmatrix} 0 & -a_5\sigma & 0 & -c_4\delta & -c_4\sigma \\ a_5\sigma & 0 & -\gamma & \delta & 0 \\ 0 & \gamma & 0 & -\sigma & 0 \\ c_4\delta & \gamma & \sigma & 0 & 0 \\ c_4\sigma & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since $\mathcal{G}$ is an MD-algebra, $rankB_F$ only get two values zero or two. Hence, it is easy to prove that $c_4 = a_5 = 0$. But this implies that $\mathcal{G}$ is decomposable. This is a contradiction. Thus, this case cannot happen.

Case 3: $e_4 \neq 0$.

By changing X_4 with $X_4' = e_4X_4 + e_3X_3$, we can assume that $e_3 = 0, e_4 = 1$. Now, Relations (II) can be rewritten as follows

$$\begin{cases} c_3 = b_4f_3, \\ b_3 = c_4. \end{cases}$$

By changing X_1 with $X_1' = X_1 - b_4X_2$, we can assume that $b_4 = 0$. Putting this into the relation above, we get $c_3 = 0$.

Then, the Lie brackets in $\mathcal{G}$ can be reduced as follows

$$\begin{aligned} [X_1, X_2] &= & & \lambda X_5, \\ [X_1, X_3] &= & \mu X_3, \\ [X_1, X_4] &= & & \mu X_4, \\ [X_1, X_5] &= & & & 2\mu X_5, \\ [X_2, X_3] &= & & X_4, \\ [X_2, X_4] &= & \theta X_3, \\ [X_3, X_4] &= & & & X_5. \end{aligned}$$

Let $F = \alpha X_1^* + \beta X_2^* + \gamma X_3^* + \delta X_4^* + \sigma X_5^* \in \mathcal{G}^*$ be an arbitrary element from $\mathcal{G}^* \equiv \mathbb{R}^5$. Then by simple computation, we see that

$$B_F = \begin{bmatrix} 0 & -\lambda\sigma & -\mu\gamma & -\mu\delta & -2\mu\sigma \\ \lambda\sigma & 0 & -\delta & -\theta\gamma & 0 \\ \mu\gamma & \delta & 0 & -\sigma & 0 \\ \mu\delta & \theta\gamma & \sigma & 0 & 0 \\ 2\mu\sigma & 0 & 0 & 0 & 0 \end{bmatrix}.$$

3.1. Assume $\mu \neq 0$.

When $\sigma \neq 0$, we have $rankB_F = 4$. Because $\mathcal{G}$ is an MD5-algebra, we get $rankB_F \in \{0, 4\}$ for all $\alpha, \beta, \delta, \gamma, \sigma \in \mathbb{R}$. But this only can happen if

$\theta < 0$. By changing X_1, X_2, X_4, X_5 with $X_1' = \mu X_1, X_2' = \sqrt{-\theta} X_2, X_4' = \sqrt{-\theta} X_4, X_5' = \sqrt{-\theta} X_5$, we can assume that $\theta = -1$. By changing X_2 with $X_2' = X_2 - \frac{\lambda}{2} X_5$, we can assume that $\lambda = 0$. Hence, the Lie brackets of $\mathcal{G}$ can be reduced as follows

$$
\begin{aligned}
[X_1; X_3] &= & X_3, \\
[X_1; X_4] &= & & X_4, \\
[X_1; X_5] &= & & & 2X_5, \\
[X_2; X_3] &= & & X_4, \\
[X_2; X_4] &= & -X_3, \\
[X_3; X_4] &= & & & X_5.
\end{aligned}
$$

3.2 Assume $\mu = 0$. By the same way, we consider the matrix of the bilinear form B_F and obtain $\lambda = 0$. But this shows that $\mathcal{G}$ is decomposable. This construction show that this case cannot happen.

The theorem 2.2.1 is proved completely. So there is only one MD5-algebra having non-commutative derived ideal. □

CONCLUDING REMARK

Let us recall that each real Lie algebra $\mathcal{G}$ defines only one connected and simply connected Lie group G such that Lie(G) = $\mathcal{G}$. Therefore we obtain only one connected and simply connected MD5-group corresponding to the MD5-algebra given in Theorem 2.2.1. In the next paper, we shall describe the geometry of K-orbits of considered MD5-group, describe topological properties of MD5-foliation formed by the generic K-orbits of this MD5-group and give the characterization of the Connes's C^*-algebra associated to this MD5-foliation.

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