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MONOIDAL FUNCTORS BETWEEN (BRAIDED) GR-CATEGORIES AND THEIR APPLICATIONS

Nguyen Tien Quang*, Nguyen Thu Thuy†
and
Pham Thi Cuc‡

* *Department of Mathematics*
Hanoi National University of Education, Hanoi, Vietnam
e-mail: cn.nguyenquang@gmail.com

† *Science Faculty, Academy of Finance*
Hanoi, Vietnam
e-mail: ntthuy11@gmail.com

‡ *Natural Science Department*
Hongduc University, Thanhhoa, Vietnam
e-mail: cucphamhd@gmail.com

Abstract

In this paper we state the basic results on Gr-functors between Gr-categories. They allow one to prove precise theorems on the classification of (braided) categorical groups and their (braided) monoidal functors. We also obtain some well-known results in algebra.

Introduction

Monoidal categories (symmetric monoidal categories) can be "refined" to become *categories with* (*abelian*) *group structure* if the objects are all *invertible* (see [8, 14]). When the underlying category is a *groupoid,* we obtain the notion of (*symmetric*) *categorical group*, or *Gr-category* (*Picard category*). The

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structures of Gr-categories, Picard categories were dealt with by H. X. Sinh in [15]. *Braided categorical groups* were originally introduced by A. Joyal and R. Street in [7] as extensions of Picard categories. The category $\mathcal{BCG}$ of braided categorical groups and braided monoidal functors was classified by the category $\mathcal{Q}uad$ of *quadratic* functions. These classification theorems were applied and extended in works on (braided) graded categorical groups [3, 5], in those on fibred categorical groups [2], and they led to many noticeable results. However, reviewing even the most basic theory of monoidal categories is interesting and useful for the further studies.

The classification of Gr-categories whose pre-stick of type (Π, A) was done by H. X. Sinh in [15], but her thesis was never published, and now quite hard to find. Also, while the results on classification of Gr-categories, braided Gr-categories were greatly clarified by A. Joyal and R. Street in [6], this section was omitted from the final published version [7]. J. Baez and A. Lauda [1] summarized in detail the results on Gr-categories, but they did not mention the classification. Our first aim is to fill this gap. In concrete, we show the results on Gr-functors and use them as a common technique to state classification theorems for categorical groups, braided categorical groups. The second aim is to show some algebraic applications of the obstruction theory of Gr-functors.

The plan of this paper is, briefly, as follows. In the first section we review the construction of a Gr-category of type (Π, A, h), a reduction of a Gr-category, due to H. X. Sinh [15].

In the second section we prove that each Gr-functor between reduced Gr-categories is one of type (φ, f). Then, we introduce the notion of obstruction of a functor of type (φ, f) and classify up to cohomology all these functors.

The next section is devoted to showing the precise classification theorem of the category of Gr-categories and Gr-functors, a fuller version of classification theorem of H. X. Sinh. The case of braided Gr-categories follows as a consequence.

Section 4 is dedicated to stating two applications of the obstruction theory of Gr-functors. Firstly, we define the Gr-category of an *abstract kernel* as an example for general theory. This leads to an interesting consequence: a Gr-category can be transformed into a strict one (H. X. Sinh proved this result in a completely different way [16]).

Secondly, we also use the Gr-category of an abstract kernel to classify group extensions by Gr-functors. Then we obtain long-known results on the group extension problem.

We should remark that Proposition 5 is used to introduce a new proof of the Classification Theorem for graded Gr-categories [12] by the method of factor sets, and a version of Proposition 5 for Ann-functors [11] is used to classify Ann-functors.

1 Preliminaries

We recall briefly some basic facts and results about monoidal categories.

A *monoidal category* $(\mathbb{G}, \otimes, I, \mathbf{a}, \mathbf{l}, \mathbf{r})$ is a category $\mathbb{G}$ together with a tensor product $\otimes : \mathbb{G} \times \mathbb{G} \to \mathbb{G}$, an *unit* object I, and natural isomorphisms

$$\mathbf{a}_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z,$$
$$\mathbf{l}_X : I \otimes X \to X\ ,\ \mathbf{r}_X : X \otimes I \to X,$$

(called, respectively, the *associativity* and *the left-unit*, and *the right-unit constraints*). These constraints satisfy the pentagon axiom

$$(\mathbf{a}_{X,Y,Z} \otimes id_T)\ \mathbf{a}_{X,Y\otimes Z,T}\ (id_X \otimes \mathbf{a}_{Y,Z,T}) = \mathbf{a}_{X\otimes Y,Z,T}\ \mathbf{a}_{X,Y,Z\otimes T}, \quad (1)$$

and the triangle axiom

$$id_X \otimes \mathbf{l}_Y = (\mathbf{r}_X \otimes id_Y)\mathbf{a}_{X,I,Y}. \quad (2)$$

A monoidal category is *strict* if the associativity constraint $\mathbf{a}$ and the unit constraints $\mathbf{l}, \mathbf{r}$ are all identities.

Let $\mathbb{G} = (\mathbb{G}, \otimes, I, \mathbf{a}, \mathbf{l}, \mathbf{r})$ and $\mathbb{G}' = (\mathbb{G}', \otimes, I', \mathbf{a}', \mathbf{l}', \mathbf{r}')$ be monoidal categories. A *monoidal functor* from $\mathbb{G}$ to $\mathbb{G}'$, $(F, \widetilde{F}, F_*)$, consists of a functor $F : \mathbb{G} \to \mathbb{G}'$, an isomorphism $F_* : I' \to FI$, and natural isomorphisms

$$\widetilde{F}_{X,Y} : FX \otimes FY \to F(X \otimes Y)$$

such that for all $X, Y, Z \in \mathbb{G}$, coherence conditions hold

$$F(\mathbf{a}_{X,Y,Z}) \circ \widetilde{F}_{X,YZ} \circ (id_{FX} \otimes \widetilde{F}_{Y,Z}) = \widetilde{F}_{X\otimes Y,Z} \circ (\widetilde{F}_{X,Y} \otimes id_{FZ}) \circ \mathbf{a}'_{FX,FY,FZ},$$

$$\mathbf{r}'_{FX} = F(\mathbf{r}_X) \circ \widetilde{F}_{X,I} \circ (id \otimes F_*) : FX \otimes I' \to FX,$$
$$\mathbf{l}'_{FX} = F(\mathbf{l}_X) \circ \widetilde{F}_{I,X} \circ (F_* \otimes id) : I' \otimes FX \to FX.$$

A *natural monoidal equivalence* or a *homotopy* $\alpha : (F, \widetilde{F}, F_*) \to (G, \widetilde{G}, G_*)$ between monoidal functors from $\mathbb{G}$ to $\mathbb{G}'$ is a natural equivalence $\alpha : F \to G$ such that

$$G_* = \alpha_I \circ F_*,$$
$$\alpha_{X\otimes Y} \circ \widetilde{F}_{X,Y} = \widetilde{G}_{X,Y} \circ (\alpha_X \otimes \alpha_Y).$$

A *monoidal equivalence* between monoidal categories is a monoidal functor $F : \mathbb{G} \to \mathbb{G}'$ such that there exists a monoidal functor $G : \mathbb{G}' \to \mathbb{G}$ and homotopies $\alpha : G.F \to id_{\mathbb{G}}$, $\beta : F.G \to id_{\mathbb{G}'}$. $(F, \widetilde{F}, F_*)$ is a monoidal equivalence if and only if F is an equivalence.

A *Gr-category* is a monoidal category in which every object is invertible and every morphism is an isomorphism. If $(F, \widetilde{F}, F_*)$ is a monoidal functor between

Gr-categories, it is called a *Gr-functor.* Then the isomorphism $F_* : I' \to FI$ can be deduced from F and $\widetilde{F}$.

Let us recall some known results on Gr-categories (see [15]). Each Gr-category $\mathbb{G}$ is equivalent to a Gr-category of type (Π, A) which can be described as follows. The set $\pi_0\mathbb{G}$ of isomorphism classes of the objects in $\mathbb{G}$ is a group where the operation is induced by the tensor product in $\mathbb{G}$, and the set $\pi_1\mathbb{G}$ of automorphisms of the unit object I is an abelian group where the operation, denoted by $+$, is composition. Moreover, $\pi_1\mathbb{G}$ is a $\pi_0\mathbb{G}$-module with the action

$$su = \gamma_X^{-1}\delta_X(u),\ X \in s,\ s \in \pi_0\mathbb{G},\ u \in \pi_1\mathbb{G},$$

where $\gamma_X,\ \delta_X$ are respectively defined by the following commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{\gamma_X(u)} & X \\ {\scriptstyle \mathbf{l}_X}\uparrow & & \uparrow{\scriptstyle \mathbf{l}_X} \\ I\otimes X & \xrightarrow{u\otimes id} & I\otimes X, \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\delta_X(u)} & X \\ {\scriptstyle \mathbf{r}_X}\uparrow & & \uparrow{\scriptstyle \mathbf{r}_X} \\ X\otimes I & \xrightarrow{id\otimes u} & X\otimes I. \end{array}$$

The *reduced* Gr-category $S_\mathbb{G}$ of $\mathbb{G}$ is a category whose objects are elements of $\pi_0\mathbb{G}$ and whose morphisms are automorphisms $(s, u) : s \to s$, where $s \in \pi_0\mathbb{G},\ u \in \pi_1\mathbb{G}$. The composition of two morphisms is induced by the addition in $\pi_1\mathbb{G}$

$$(s, u).(s, v) = (s, u + v).$$

The category $S_\mathbb{G}$ is equivalent to $\mathbb{G}$ by canonical equivalences constructed as follows. For each $s = [X] \in \pi_0\mathbb{G}$, choose a representative $X_s \in \mathbb{G}$ such that $X_1 = I$, and for each $X \in s$, choose an isomorphism $i_X :\ X_s \to X$ such that $i_{X_s} = id_{X_s}$. For the set of representatives, we obtain two functors

$$\begin{cases} G : \mathbb{G} \to S_\mathbb{G}, \\ G(X) = [X] = s, \\ G(X \xrightarrow{f} Y) = (s, \gamma_{X_s}^{-1}(i_Y^{-1} f i_X)), \end{cases} \qquad \begin{cases} H : S_\mathbb{G} \to \mathbb{G}, \\ H(s) = X_s, \\ H(s, u) = \gamma_{X_s}(u). \end{cases} \tag{3}$$

Two functors G and H are categorical equivalences by natural transformations

$$\alpha = (i_X) : HG \cong id_\mathbb{G}, \qquad \beta = id : GH \cong id_{S_\mathbb{G}}.$$

They are called *canonical equivalences*.

Via the structure transport (see [12, 15]) by the quadruple (G, H, α, β), $S_\mathbb{G}$ becomes a Gr-category together with the following operations

$$s \otimes t = st, \quad s, t \in \pi_0\mathbb{G},$$
$$(s, u) \otimes (t, v) = (st, u + sv), \quad u, v \in \pi_1\mathbb{G}.$$

The set of representatives (X_s, i_X) is a *stick* of the Gr-category $\mathbb{G}$ whenever

$$i_{I\otimes X_s} = \mathbf{l}_{X_s},\ i_{X_s\otimes I} = \mathbf{r}_{X_s}. \tag{4}$$

The unit constraints of the Gr-category $S_{\mathbb{G}}$ are therefore strict, and its associativity constraint is a normalized 3-cocycle $h \in Z^3(\pi_0\mathbb{G}, \pi_1\mathbb{G})$. Further, the equivalences $G,\ H$ together with natural isomorphisms

$$\widetilde{G}_{X,Y} = G(i_X \otimes i_Y),\ \widetilde{H}_{s,t} = i^{-1}_{X_s\otimes X_t} \tag{5}$$

become Gr-equivalences.

The Gr-category $S_{\mathbb{G}}$ is a *reduction* of the Gr-category $\mathbb{G}$. $S_{\mathbb{G}}$ is said to be of *type* (Π, A, h), or of *type* (Π, A) if $\pi_0\mathbb{G}, \pi_1\mathbb{G}$ are replaced with a group Π and a Π-module A, respectively.

2 Classification of Gr-functors of type (φ, f)

In this section, we show that each Gr-functor $(F, \widetilde{F}) : \mathbb{G} \to \mathbb{G}^{'}$ induces a Gr-functor S_F between the reduced Gr-categories. This allows us to study the existence of Gr-functors and to classify all these functors between Gr-categories of type (Π, A). The following proposition appeared in many works related to categorical groups.

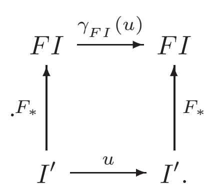
Proposition 1 ([15]). *Let $(F, \widetilde{F}) : \mathbb{G} \to \mathbb{G}'$ be a Gr-functor. Then, $(F, \widetilde{F})$ induces a pair of group homomorphisms*

$$F_0 : \pi_0\mathbb{G} \to \pi_0\mathbb{G}^{'},\quad [X] \mapsto [FX],$$
$$F_1 : \pi_1\mathbb{G} \to \pi_1\mathbb{G}^{'},\quad u \mapsto \gamma^{-1}_{FI}(Fu)$$

satisfying $F_1(su) = F_0(s)F_1(u)$.

Our first result is to strengthen Proposition 1 by Proposition 4 thanks to the fact that each Gr-functor $(F, \widetilde{F}) : \mathbb{G} \to \mathbb{G}'$ induces a Gr-functor $S_F : S_{\mathbb{G}} \to S_{\mathbb{G}'}$. We need two following lemmas

Lemma 2. *Let $\mathbb{G}, \mathbb{G}'$ be two $\otimes$-categories with, respectively, unit constraints $(I, \mathbf{l}, \mathbf{r})$, $(I', \mathbf{l}', \mathbf{r}')$, and $(F, \widetilde{F}, F_*) : \mathbb{G} \to \mathbb{G}'$ be an $\otimes$-functor which is compatible with the unit constraints. Then, the following diagram commutes*



It follows that

$$\gamma_{FI}^{-1}(Fu) = F_*^{-1}F(u)F_*,$$

i.e., the definitions of the map $\pi_1 F$ in [3] and of the map F_1 in Proposition 1 coincide.

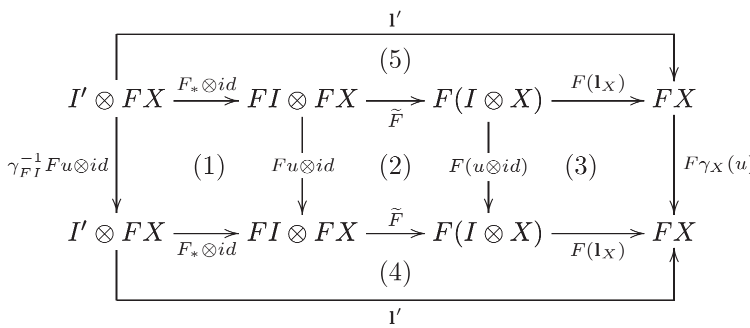
Proof. Clearly, $\gamma_{I'}(u) = u$. Moreover, the family $(\gamma_{X'}(u))$, $X' \in \mathbb{G}'$, is an endomorphism of the identity functor $id_{\mathbb{G}'}$. So the above diagram commutes.

The final statement is deduced from the above commutative diagram in which u is replaced by $\gamma_{FI}^{-1}(Fu)$. □

Lemma 3. *Under the hypothesis of Lemma 2, we have*

$$F\gamma_X(u) = \gamma_{FX}(\gamma_{FI}^{-1}Fu).$$

Proof. Consider the following diagram



In this diagram, the regions (4) and (5) commute thanks to the compatibility of the functor $(F, \widetilde{F})$ with the unit constraints. The region (3) commutes due to the definition of γ_X (taking images via F), the region (1) commutes by Lemma 2. The commutativity of the region (2) follows from the naturality of the isomorphism $\widetilde{F}$. Therefore, the outer perimeter commutes, i.e., $F\gamma_X(u) = \gamma_{FX}\big(\gamma_{FI}^{-1}Fu\big)$. □

Remark on notations: Hereafter, if there is no extra words, $\mathbb{S}$ and $\mathbb{S}'$ are referred to Gr-categories of type (Π, A, h) and (Π', A', h'), respectively.

A functor $F : \mathbb{S} \to \mathbb{S}'$ is called a functor of *type* (φ, f) if

$$F(x) = \varphi(x), \quad F(x, u) = (\varphi(x), f(u)), \tag{6}$$

where $\varphi : \Pi \to \Pi'$, $f : A \to A'$ are group homomorphisms satisfying $f(xa) = \varphi(x)f(a)$ for $x \in \Pi, a \in A$.

Proposition 4. *Each Gr-functor $(F, \widetilde{F}) : \mathbb{G} \to \mathbb{G}'$ induces a Gr-functor $S_F : S_{\mathbb{G}} \to S_{\mathbb{G}'}$ of type (φ, f), $\varphi = F_0, f = F_1$. Further,*

$$S_F = G'FH,$$

where H, G' are canonical equivalences.

Proof. Let $K = G'FH$ be the composition functor. One can verify that $K(s) = F_0(s)$, for $s \in \pi_0\mathbb{G}$. We now prove that $K(s,u) = (F_0(s), F_1(u))$ for any morphism $u : I \to I$. We have

$$K(s,u) = G'FH(s,u) = G'(F\gamma_{X_s}(u)).$$

Since $H'G' \simeq id_{\mathbb{G}'}$ by the natural equivalence $\alpha' = (i'_{X'})$, the following diagram commutes (note that $X'_s = H'G'FX_s$)

$$\begin{array}{ccc} X'_{s'} & \xrightarrow{i'} & FX_s \\ {\scriptstyle H'G'F\gamma_{X_s}(u)}\downarrow & & \downarrow{\scriptstyle F\gamma_{X_s}(u)} \\ X'_{s'} & \xrightarrow{i'} & FX_s. \end{array}$$

According to Lemma 3, we have

$$F\gamma_{X_s}(u) = \gamma_{FX_s}(\gamma_{FI}^{-1}F(u)).$$

Besides, since the family $(\gamma_{X'})$ is a natural equivalence of the identity functor $id_{\mathbb{G}'}$, the following diagram commutes

$$\begin{array}{ccc} X'_{s'} & \xrightarrow{i'} & FX_s \\ {\scriptstyle \gamma_{X'_s}(\gamma_{FI}^{-1}Fu)}\downarrow & & \downarrow{\scriptstyle \gamma_{FX_s}(\gamma_{FI}^{-1}F(u))} \\ X'_{s'} & \xrightarrow{i'} & FX_s. \end{array}$$

Hence, $H'G'F\gamma_{X_s}(u) = \gamma_{X'_s}\big(\gamma_{FI}^{-1}F(u)\big)$. By the definition of H', we have

$$G'F\gamma_{X_s}(u) = (F_0(s), \gamma_{FI}^{-1}F(u)) = (F_0(s), F_1(u)).$$

This means $K = S_F$. □

We now describe Gr-functors between Gr-categories of type (Π, A).

Proposition 5. *Every Gr-functor $(F, \widetilde{F}) : \mathbb{S} \to \mathbb{S}'$ is a functor of type (φ, f).*

Proof. For $x, y \in \Pi$, $\widetilde{F}_{x,y} : Fx \otimes Fy \to F(x \otimes y)$ is a morphism in $\mathbb{S}'$. It follows that $Fx.Fy = F(xy)$. So if one sets $\varphi(x) = Fx$, then $\varphi : \Pi \to \Pi'$ is a group homomorphism.

We write $F(x,a) = (\varphi(x), f_x(a))$. Since F is a functor, we have

$$F((x,a).(x,b)) = F(x,a).F(x,b).$$

It follows that

$$f_x(a+b) = f_x(a) + f_x(b).$$

Therefore, $f_x : A \to A'$ is a group homomorphism for each $x \in \Pi$. Besides, since $(F, \widetilde{F})$ is an $\otimes$-functor, the following diagram commutes

$$\begin{array}{ccc} Fx.Fy & \xrightarrow{\widetilde{F}} & F(xy) \\ {\scriptstyle Fu\otimes Fv}\big\downarrow & & \big\downarrow{\scriptstyle F(u\otimes v)} \\ Fx.Fy & \xrightarrow{\widetilde{F}} & F(xy), \end{array}$$

for all $u = (x, a),\ v = (y, b)$. Hence, we have

$$\begin{aligned} &F(u \otimes v) = Fu \otimes Fv \\ &\Leftrightarrow f_{xy}(a + xb) = f_x(a) + \varphi(x).f_y(b) \\ &\Leftrightarrow\ f_{xy}(a) + f_{xy}(xb) = f_x(a) + \varphi(x).f_y(b). \end{aligned} \tag{7}$$

In the relation (7), $x = 1$ yields $f_y(a) = f_1(a)$. Hence, $f_y = f_1$ for all $y \in \Pi$. Write $f_y = f$ and use (7), we obtain $f(xb) = \varphi(x).f(b)$. □

Note that if Π'-module A' is regarded as a Π-module with the action $xa' = \varphi(x).a'$, then $f : A \to A'$ is a homomorphism of Π-modules. Since

$$\widetilde{F}_{x,y} = (F(xy), g_F(x, y)) : Fx.Fy \to F(xy),$$

where $g_F : \Pi^2 \to A'$ is a function, g_F is said to be *associated* to $\widetilde{F}$. The compatibility of $(F, \widetilde{F})$ with the associativity constraint leads to the relation

$$\varphi^* h' - f_* h = \partial(g_F),$$

where

$$\begin{aligned} (f_* h)(x, y, z) &= f(h(x, y, z)), \\ (\varphi^* h')(x, y, z) &= h'(\varphi x, \varphi y, \varphi z). \end{aligned}$$

One can see that two Gr-functors $(F, \widetilde{F}), (F', \widetilde{F}') : \mathbb{S} \to \mathbb{S}'$ are homotopic if and only if $F' = F$, i.e., they are of the same type (φ, f), and there is a function $t : \Pi \to A'$ such that $g_{F'} = g_F + \partial t$.

We refer to

$$\mathrm{Hom}_{(\varphi,f)}[\mathbb{S}, \mathbb{S}'].$$

as the set of homotopy classes of Gr-functors of type (φ, f) from S to S.

In order to find the sufficient condition to make a functor of type (φ, f) become a Gr-functor, we introduce the notion of *the obstruction* as in the case of Ann-functors (see [11]). If h and h' are associativity constraints of Gr-categories $\mathbb{S}$ and $\mathbb{S}'$, respectively, and $F : \mathbb{S} \to \mathbb{S}'$ is a functor of type (φ, f), then the function

$$k = \varphi^* h' - f_* h \tag{8}$$

is called *an obstruction* of F.

Keeping in mind that $\mathbb{S} = (\Pi, A, h),\ \mathbb{S}' = (\Pi', A', h')$, we have

Theorem 6. *The functor $F : \mathbb{S} \to \mathbb{S}'$ of type (φ, f) induces a Gr-functor if and only if its obstruction $[k] = 0$ in $H^3(\Pi, A')$. Then, there exist bijections*

$$\mathrm{Hom}_{(\varphi,f)}[\mathbb{S}, \mathbb{S}'] \leftrightarrow H^2(\Pi, A'), \tag{9}$$

$$\mathrm{Aut}(F) \leftrightarrow Z^1(\Pi, A').$$

Proof. If $(F, \widetilde{F}) : \mathbb{S} \to \mathbb{S}'$ is a Gr-functor, then $(F, \widetilde{F}) = (\varphi, f, g_F)$, where

$$\varphi^* h' - f_* h = \partial(g_F) \in B^3(\Pi, A').$$

Therefore, $[\varphi^* h'] - [f_* h] = 0$ in $H^3(\Pi, A')$.

Conversely, if $[\varphi^* h'] - [f_* h] = 0$, then there exists a 2-cochain $g \in Z^2(\Pi, A')$ such that $\varphi^* h' - f_* h = \partial g$. Take $\widetilde{F}$ to be associated to g, one can see that $(F, \widetilde{F})$ is a Gr-functor.

i) If $(F, \widetilde{F}) : \mathbb{S} \to \mathbb{S}'$ is a Gr-functor, then $F = (\varphi, f, g_F)$. Let g_F be fixed. Now if

$$(K, \widetilde{K}) : \mathbb{S} \to \mathbb{S}'$$

is a Gr-functor of type (φ, f), then $\partial(g_F) = \varphi^* h' - f_* h = \partial(g_K)$. It follows that $g_F - g_K$ is a 2-cocycle. Consider the correspondence

$$\Phi : [(K, \widetilde{K})] \mapsto [g_F - g_K]$$

between the set of equivalent classes of Gr-functors of type (φ, f) from $\mathbb{S}$ to $\mathbb{S}'$ and the group $H^2(\Pi, A')$.

First, we show that the above correspondence is a map. Indeed, let

$$(K', \widetilde{K'}) : \mathbb{S} \to \mathbb{S}'$$

be a Gr-functor and $u : K \to K'$ be a homotopy. Then K, K' are of the same type (φ, f) and $g_{K'} = g_K + \partial t$ where $g_K, g_{K'}$ are, respectively, associated to $\widetilde{K}, \widetilde{K'}$, i.e., $[g_F - g_{K'}] = [g_F - g_K] \in H^2(\Pi, A')$.

Furthermore, Φ is an injection.

Finally, we show that the correspondence Φ is a surjection. Assume that g is an arbitrary 2-cocycle, we have

$$\partial(g_F - g) = \partial g_F - \partial g = \partial g = \varphi^* h' - f_* h.$$

Then, there exists a Gr-functor

$$(K, \widetilde{K}) : \mathbb{S} \to \mathbb{S}'$$

of type (φ, f) where natural isomorphisms $\widetilde{K} = (\bullet, g_F - g)$. So Φ is a surjection.

ii) Let $F = (F, \widetilde{F}) : \mathbb{S} \to \mathbb{S}'$ be a Gr-functor and $t \in \mathrm{Aut}(F)$. Then, the equality $g_F = g_F + \partial t$ implies that $\partial t = 0$, i.e., $t \in Z^1(\Pi, A')$. □

3 Classification of (braided) Gr-categories

Each Gr-category $\mathbb{G}$ has two first invariants as the group $\pi_0\mathbb{G}$ and $\pi_0\mathbb{G}$-module $\pi_1\mathbb{G}$. The set of Gr-categories having the same two first invariants was classified by the cohomology group $H^3(\pi_0\mathbb{G}, \pi_1\mathbb{G})$ (see [15]). Now, we will state the precise theorem on classification of Gr-categories and Gr-functors.

Let $\mathcal{CG}$ be a category whose objects are Gr-categories, and whose morphisms are monoidal functors between them. We determine the category $\mathbf{H^3_{Gr}}$ whose objects are triples $(\Pi, A, [h])$ where $[h] \in H^3(\Pi, A)$. A morphism $(\varphi, f) : (\Pi, A, [h]) \to (\Pi', A', [h'])$ in $\mathbf{H^3_{Gr}}$ is a pair (φ, f) such that there is a function $g : \Pi^2 \to A'$ so that $(\varphi, f, g) : (\Pi, A, h) \to (\Pi', A', h')$ is a Gr-functor, i.e., $[\varphi^* h'] = [f_* h] \in H^3(\Pi, A')$. The composition in $\mathbf{H^3_{Gr}}$ is given by

$$(\varphi', f') \circ (\varphi, f) = (\varphi'\varphi, f'f).$$

One can observe that *two Gr-functors* $F, F' : \mathbb{G} \to \mathbb{G}'$ *are homotopic if and only if* $F_i = F'_i, i = 0, 1,$ *and* $[g_F] = [g_{F'}]$. Denote the set of homotopy classes of Gr-functors $\mathbb{G} \to \mathbb{G}'$ which induce the same pair (φ, f) by

$$\mathrm{Hom}_{(\varphi,f)}[\mathbb{G}, \mathbb{G}'].$$

We now state a version of Proposition 8 [6], in which the classifying functor d is not simply "inverse" of that in Proposition 8 [6], but it contains more information in this classification.

Theorem 7 (Classification Theorem). *There is a classifying functor*

$$\begin{array}{rccc} d: & \mathcal{CG} & \to & \mathbf{H^3_{Gr}} \\ & \mathbb{G} & \mapsto & (\pi_0\mathbb{G}, \pi_1\mathbb{G}, [h_{\mathbb{G}}]) \\ & (F, \widetilde{F}) & \mapsto & (F_0, F_1) \end{array}$$

which has the following properties

i) dF *is an isomorphism if and only if* F *is an equivalence.*

ii) d *is surjective on objects.*

iii) d *is full, but not faithful. For* $(\varphi, f) : d\mathbb{G} \to d\mathbb{G}'$*, there is a bijection*

$$\overline{d} : Hom_{(\varphi,f)}[\mathbb{G}, \mathbb{G}'] \to H^2(\pi_0\mathbb{G}, \pi_1\mathbb{G}'). \tag{10}$$

Proof. In the Gr-category $\mathbb{G}$, for each stick (X_s, i_X) we can construct a reduced Gr-category $(\pi_0\mathbb{G}, \pi_1\mathbb{G}, h)$. If the choice of stick is modified, then the 3-cocycle h changes to a cohomologous 3-cocycle h'. Therefore, $\mathbb{G}$ determines a unique element $[h] \in H^3(\pi_0\mathbb{G}, \pi_1\mathbb{G})$. This shows that d is a map on objects.

For Gr-functors

$$\mathbb{G} \xrightarrow{F} \mathbb{G}' \xrightarrow{F'} \mathbb{G}'',$$

one can see that $(F'F)_0 = F'_0F_0$. Since $(F'F)_*$ is the composition

$$I'' \overset{F'_*}{\to} F'I' \overset{F'(F_*)}{\to} F'FI,$$

then for $u \in \mathrm{Aut}(I)$ we have

$$(F'F)_1 u = (F'F)_*^{-1}(F'F)u(F'F)_*$$
$$= F_*'^{-1}F'(F_*^{-1})F'FuF'(F_*)F'_*$$
$$= F_*'^{-1}F'(F_1u)F'_* = F'_1(F_1u).$$

That is,

$$d(F' \circ F) = (dF') \circ (dF).$$

Clearly, $d(id_{\mathbb{G}}) = id_{d\mathbb{G}}$. Therefore, d is a functor.
i) According to Proposition 1.
ii) If $(\Pi, A, [h])$ is an object of $\mathbf{H^3_{Gr}}$, $\mathbb{S} = (\Pi, A, h)$ is a Gr-category of type (Π, A) and obviously $d\mathbb{S} = (\Pi, A, [h])$.
iii) Let (φ, f) be a morphism in $\mathrm{Hom}_{\mathbf{H^3_{Gr}}}(d\mathbb{G}, d\mathbb{G}')$. Then, there exists a function $g : (\pi_0\mathbb{G})^2 \to \pi_1\mathbb{G}'$ such that

$$\varphi^* h_{\mathbb{G}'} = f_* h_{\mathbb{G}} + \partial g.$$

Hence, by Theorem 6,

$$K = (\varphi, f, g) : (\pi_0\mathbb{G}, \pi_1\mathbb{G}, h_{\mathbb{G}}) \to (\pi_0\mathbb{G}', \pi_1\mathbb{G}', h_{\mathbb{G}'})$$

is a Gr-functor. Then, the composition Gr-functor $F = H'KG : \mathbb{G} \to \mathbb{G}'$ induces $dF = (\varphi, f)$. This shows that the functor d is full.

To prove that (10) is a bijection, we prove the correspondence

$$\Omega : \mathrm{Hom}_{(\varphi,f)}[\mathbb{G}, \mathbb{G}'] \to \mathrm{Hom}_{(\varphi,f)}[S_{\mathbb{G}}, S_{\mathbb{G}'}] \qquad (11)$$

$$[F] \mapsto [S_F]$$

is a bijection.

Clearly, if $F, F' : \mathbb{G} \to \mathbb{G}'$ are homotopic, then the induced Gr-functors $S_F, S_{F'} : S_{\mathbb{G}} \to S_{\mathbb{G}'}$ are homotopic. Conversely, if F, F' are Gr-functors such that $S_F, S_{F'}$ are homotopic, then the compositions $E = H'(S_F)G$ and $E' = H'(S_{F'})G$ are homotopic, where H', G are canonical Gr-equivalences. The Gr-functors E and E' are homotopic to F and F', respectively. Hence, F and F' are homotopic. This shows that Ω is an injection.

Now, if $K = (\varphi, f, g) : S_{\mathbb{G}} \to S_{\mathbb{G}'}$ is a Gr-functor, then the composition

$$F = H'KG : \mathbb{G} \to \mathbb{G}'$$

is a Gr-functor satisfying $S_F = K$, i.e., Ω is a surjection. Finally, the bijection (10) is the composition of (9) and (11). $\square$

By Theorem 7, one can simplify the problem of classifying of Gr-categories up to equivalence by the one of Gr-categories having the same (up to an isomorphism) two first invariants. This was done by H. X. Sinh [15]. However, to make it easy for the readers we will state this result in detail based on the above data.

Let Π be a group and A be a Π-module. We say that a Gr-category $\mathbb{G}$ has a *pre-stick of type* (Π, A) if there exists a pair of group isomorphisms

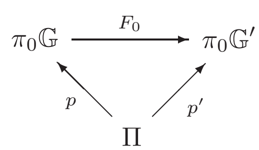
$$p : \Pi \to \pi_0\mathbb{G}, \quad q : A \to \pi_1\mathbb{G}$$

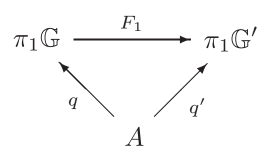
which are compatible with the action of modules

$$q(su) = p(s)q(u),$$

where $s \in \Pi, u \in A$. The pair $\epsilon = (p, q)$ is called a *pre-stick of type* (Π, A) to the Gr-category $\mathbb{G}$.

A *morphism* between two Gr-categories $\mathbb{G}, \mathbb{G}'$ whose pre-sticks are of type (Π, A) (with, respectively, the pre-sticks $\epsilon = (p, q), \epsilon' = (p', q')$) is a Gr-functor $(F, \widetilde{F}) : \mathbb{G} \to \mathbb{G}'$ such that the following diagrams commute





where F_0, F_1 are two homomorphisms induced by $(F, \widetilde{F})$.

Clearly, it follows from the definition that F_0, F_1 are isomorphisms and therefore F is an equivalence. Denote by

$$\mathcal{CG}[\Pi, A]$$

the set of equivalence classes of Gr-categories whose pre-sticks are of type (Π, A).

Theorem 8 ([15]). *There exists a bijection*

$$\Gamma : \mathcal{CG}[\Pi, A] \to H^3(\Pi, A),$$

$$[\mathbb{G}] \mapsto q_*^{-1}p^*[h_{\mathbb{G}}].$$

Proof. By Theorem 7 , each Gr-category $\mathbb{G}$ determines uniquely an element $[h_{\mathbb{G}}] \in H^3(\pi_0\mathbb{G}, \pi_1\mathbb{G})$, and therefore

$$\epsilon[h_{\mathbb{G}}] = q_*^{-1}p^*[h_{\mathbb{G}}] \in H^3(\Pi, A).$$

Now, if $F:\mathbb{G}\to\mathbb{G}'$ is a morphism between two Gr-categories whose pre-sticks of type (Π, A), then the induced Gr-functor $S_F=(\varphi, f, g_F)$ satisfies

$$\varphi^*[h_{\mathbb{G}'}]=f_*[h_{\mathbb{G}}].$$

It follows that

$$\epsilon'[h_{\mathbb{G}'}]=\epsilon[h_{\mathbb{G}}].$$

This means Γ is a map. Moreover, it is an injection. Indeed, suppose that $\Gamma[\mathbb{G}]=\Gamma[\mathbb{G}']$, we have

$$\epsilon'(h_{\mathbb{G}'})-\epsilon(h_{\mathbb{G}})=\partial g.$$

Therefore, there exists a Gr-functor J of type (id, id) from $\mathbb{J}=(\Pi, A, \epsilon(h_{\mathbb{G}}))$ to $\mathbb{J}'=(\Pi, A, \epsilon'(h_{\mathbb{G}'}))$. The composition

$$\mathbb{G}\xrightarrow{G}S_{\mathbb{G}}\xrightarrow{\varepsilon^{-1}}\mathbb{J}\xrightarrow{J}\mathbb{J}'\xrightarrow{\varepsilon'}S_{\mathbb{G}'}\xrightarrow{H'}\mathbb{G}'$$

implies $[\mathbb{G}]=[\mathbb{G}']$, and Γ is an injection. Obviously, Γ is surjective. □

We now move to the case of braided Gr-categories.

A Gr-category $\mathbb{B}$ is called a *braided Gr-category* if there is a *braiding* $\mathbf{c}$, i.e., natural isomorphisms $\mathbf{c}=\mathbf{c}_{X,Y}:X\otimes Y\to Y\otimes X$, which is compatible with $\mathbf{a},\mathbf{l},\mathbf{r}$ in the sense of satisfying the following coherence conditions:

$$(id_Y\otimes\mathbf{c}_{X,Z})\mathbf{a}_{Y,X,Z}(\mathbf{c}_{X,Y}\otimes id_Z)=\mathbf{a}_{Y,Z,X}\mathbf{c}_{X,Y\otimes Z}\mathbf{a}_{X,Y,Z}, \tag{12}$$

$$(\mathbf{c}_{X,Z}\otimes id_Y)\mathbf{a}^{-1}_{X,Z,Y}(id_X\otimes\mathbf{c}_{Y,Z})=\mathbf{a}^{-1}_{Z,X,Y}\mathbf{c}_{X\otimes Y,Z}\mathbf{a}^{-1}_{X,Y,Z}. \tag{13}$$

If the braiding $\mathbf{c}$ satisfies $\mathbf{c}_{X,Y}\cdot\mathbf{c}_{Y,X}=id$ then braided Gr-categories are called *symmetric categorical groups*, or *Picard categories*. Then the relations (13) and (12) coincide.

If $(\mathbb{B},\mathbf{c})$ and $(\mathbb{B}',\mathbf{c}')$ are braided Gr-categories, then a braided Gr-functor $(F,\widetilde{F}):\mathbb{B}\to\mathbb{B}'$ is a Gr-functor which is compatible with the braidings $\mathbf{c},\mathbf{c}'$, i. e., the following diagram commutes

$$\begin{array}{ccc}
F(X\otimes Y) & \xrightarrow{F(\mathbf{c})} & F(Y\otimes X)\\
\uparrow{\scriptstyle \widetilde{F}} & & \uparrow{\scriptstyle \widetilde{F}}\\
FX\otimes FY & \xrightarrow{\mathbf{c}'} & FY\otimes FX.
\end{array}$$

If $\mathbb{B}$ is a braided Gr-category with the braiding $\mathbf{c}$, then $\pi_0\mathbb{B}$ is an abelian group and it acts trivially on $\pi_1\mathbb{B}$. Then the reduced Gr-category $S_{\mathbb{B}}$ becomes a braided Gr-category with the induced braiding $\mathbf{c}^* = (\bullet, \eta)$ given by the following commutative diagram

$$\begin{array}{ccc} X_r \otimes X_s & \xrightarrow{i_{X_r \otimes X_s}} & X_{rs} \\ \downarrow{\scriptstyle \mathbf{c}} & & \downarrow{\scriptstyle \gamma_{X_{rs}}(\eta(r,s))} \\ X_s \otimes X_r & \xrightarrow{i_{X_s \otimes X_r}} & X_{sr}. \end{array}$$

Moreover, $(H, \widetilde{H})$ and $(G, \widetilde{G})$ defined by (3) and by (5) are then braided Gr-equivalences.

For the reduced braided Gr-category $S_{\mathbb{B}}$, the relations (1), (12), (13) become

$$h(y,z,t) - h(x+y,z,t) + h(x,y+z,t) - h(x,y,z+t) + h(x,y,z) = 0,$$

$$h(x,y,z) - h(y,x,z) + h(y,z,x) + \eta(x,y+z) - \eta(x,y) - \eta(x,z) = 0,$$

$$h(x,y,z) - h(x,z,y) + h(z,x,y) - \eta(x+y,z) + \eta(y,z) + \eta(x,z) = 0,$$

where the functions h, η are associated to the associativity, braiding constraints of $S_{\mathbb{B}}$, respectively. By the compatibility of the associativity constraint with the strict unit ones of $S_{\mathbb{B}}$, h, η are the “normalized” functions. Therefore, the pair (h, η) associated to the associativity and braiding constraints of $S_{\mathbb{B}}$ is an abelian 3-cocycle (in the sense of Mac Lane-Eilenberg [4, 9]).

In $\mathbb{B}$, choose the stick (X_i', i_X') instead of (X_i, i_X) (see (4)) then the corresponding 3-cocycle (h', η') satisfies

$$(h', \eta') - (h, \eta) = \delta g,$$

where 3-coboundary δg is given by

$$\delta g(x,y,z) = g(y,z) - g(x+y,z) + g(x,y+z) - g(x,y),$$

$$\delta g(x,y) = g(x,y) - g(y,x).$$

This mean each braided Gr-category $\mathbb{B}$ determines uniquely an element $[(h,\eta)] \in H^3_{ab}(\pi_0\mathbb{B}, \pi_1\mathbb{B})$.

It follows from Proposition 5 that

Corollary 9. *Each braided Gr-functor* $(F, \widetilde{F}) : \mathbb{S} \to \mathbb{S}'$ *is a triple* (φ, f, g), *where*

$$\varphi^*(h', \eta') - f_*(h, \eta) = \partial_{ab}(g).$$

Based on the above data, let

$$\mathbf{H^3_{BGr}}$$

denote a category whose objects are triples $(M, N, [(h,\eta)])$, where $[(h,\eta)] \in H^3_{ab}(M, N)$. A morphism $(\varphi, f) : (M, N, [(h,\eta)]) \to (M', N', [(h',\eta')])$ in $\mathbf{H^3_{BGr}}$

is a pair (φ, f) such that there is a function $g : M^2 \to N'$ so that $(\varphi, f, g) : (M, N, (h, \eta)) \to (M', N', (h', \eta'))$ is a braided monoidal functor, i.e., $[\varphi^*(h', \eta')] = [f_*(h, \eta)] \in H^3_{ab}(M, N')$.

We write

$$\mathrm{Hom}^{Br}_{(\varphi,f)}[\mathbb{B}, \mathbb{B}']$$

for the set of homotopy classes of braided Gr-functors $\mathbb{B} \to \mathbb{B}'$ inducing the same pair (φ, f), and $\mathcal{BCG}$ for the category whose objects are braided categorical groups and whose morphisms are braided monoidal functors. Now, we state the classification theorem whose proof follows from Corollary 9, the proofs of Theorem 7 and of Theorem 8 with some suitable modifications. This theorem is aversion of Proposition 14 [6] with some modifications like Theorem 7.

Theorem 10 (Classification Theorem). *There is a classifying functor*

$$\begin{array}{rccc} d: & \mathcal{BCG} & \to & \mathbf{H^3_{BGr}}, \\ & \mathbb{B} & \mapsto & (\pi_0\mathbb{B}, \pi_1\mathbb{B}, [(h,\eta)_{\mathbb{B}}]) \\ & (F, \widetilde{F}) & \mapsto & (F_0, F_1) \end{array}$$

which has the following properties

i) *dF is an isomorphism if and only if F is an equivalence,*

ii) *d is surjective on objects,*

iii) *d is full, but not faithful. For $(\varphi, f) : d\mathbb{B} \to d\mathbb{B}'$, there is a bijection*

$$\mathrm{Hom}^{Br}_{(\varphi,f)}[\mathbb{B}, \mathbb{B}'] \cong H^2_{ab}(\pi_0\mathbb{B}, \pi_1\mathbb{B}').$$

We write

$$\mathcal{BCG}[M, N]$$

for the set of equivalence classes of braided Gr-categories whose pre-sticks are of type (M, N). By Corollary 9, we can prove the following proposition

Theorem 11. *There exists a bijection*

$$\Gamma : \mathcal{BCG}[M, N] \to H^3_{ab}(M, N),$$

$$[\mathbb{B}] \mapsto q_*^{-1} p^*[(h, \eta)_{\mathbb{B}}].$$

We complete this section by a discussion of the classification result of A. Joyal and R. Street. In [7], they classified braided Gr-categories by the *quadratic* functions as follows.

A function $\nu : M \to N$ between abelian groups is a *quadratic map* when the function $M \times M \to N, (x, y) \mapsto \nu(x) + \nu(y) - \nu(x + y)$ is bilinear and $\nu(-x) = \nu(x)$.

The *trace* of an abelian 3-cocycle $(h, \eta) \in Z^3_{ab}(M, N)$ is a function

$$t_\eta : M \to N,\ t_\eta(x) = \eta(x, x).$$

A simple calculation shows that traces are quadratic maps, and Eilenberg - MacLane [4, 9] proved that the traces determine an isomorphism

$$H^3_{ab}(M,N) \cong \mathcal{Q}uad(M,N),\ [(h,\eta)] \mapsto t_\eta, \tag{14}$$

where $\mathcal{Q}uad(M,N)$ is the abelian group of quadratic maps from M to N. This plays a fundamental role in the proof of Classification Theorem (Theorem 3.3 [7]).

A. Joyal and R. Street proved that each braided Gr-category $\mathbb{B}$ determines a quadratic function $q_{\mathbb{B}} : \pi_0\mathbb{B} \to \pi_1\mathbb{B}$ and let $\mathcal{Q}uad$ be a category whose objects (M,N,t) are quadratic maps, $t : M \to N$, between abelian groups M, N and whose morphisms $(\varphi, f) : (M,N,t) \to (M',N',t')$ consist of homomorphisms φ, f such that we have a commutative square

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & M' \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t'} \\ N & \xrightarrow{f} & N'. \end{array}$$

Theorem 12 (Theorem 3.3 [7]). *The functor*

$$\begin{array}{rccc} T: & \mathcal{BCG} & \to & \mathcal{Q}uad \\ & \mathbb{B} & \mapsto & (\pi_0\mathbb{B}, \pi_1\mathbb{B}, q_{\mathbb{B}}) \end{array}$$

has the following properties

i) *For each object Q of $\mathcal{Q}uad$, there exists an object $\mathbb{B}$ of $\mathcal{BCG}$ with an isomorphism $T(\mathbb{B}) \cong Q$;*

ii) *For any isomorphism $\rho : T(\mathbb{B}) \xrightarrow{\sim} T(\mathbb{B}')$, there is an equivalence $F : \mathbb{B} \to \mathbb{B}'$ such that $T(F) = \rho$; and*

iii) *$T(F)$ is an isomorphism if and only if F is an equivalence.*

We see that the isomorphism (14) induces an isomorphism between braided Gr-categories

$$\mathcal{V} : \mathbf{H^3_{BGr}} \to \mathcal{Q}uad$$

$$(M,N,[h,\eta]) \mapsto (M,N,t_\eta)$$

$$(\varphi, f) \mapsto (\varphi, f).$$

Hence, T is the composition

$$\mathcal{BCG} \xrightarrow{d} \mathbf{H^3_{BGr}} \xrightarrow{\mathcal{V}} \mathcal{Q}uad$$

4 Applications

4.1 Gr-category of an abstract kernel

The notion of *abstract kernel* was introduced in [10]. It is a triple (Π, G, ψ), where $\psi : \Pi \to \mathrm{Aut}G/\mathrm{In}G$ is a group homomorphism. In this section, we describe the Gr-category structure of an abstract kernel and use it to make constraints of a Gr-category be strict. The operation of G is denoted by $+$. The *center* of G, denoted by ZG, consists of elements $c \in G$ such that $c + a = a + c$ for all $a \in G$.

Let us recall that the obstruction of (Π, G, ψ) is an element $[k] \in H^3(\Pi, ZG)$ defined as follows. For each $x \in \Pi$, choose $\varphi(x) \in \psi(x)$ such that $\varphi(1) = id_G$. Then, there is a function $f : \Pi^2 \to G$ satisfying

$$\varphi(x)\varphi(y) = \mu_{f(x,y)}\varphi(xy), \tag{15}$$

where μ_c is an inner-automorphism of the group G induced by $c \in G$. The pair (φ, f) therefore induces an element $k \in Z^3(\Pi, ZG)$ defined by the relation

$$\varphi(x)[f(y,z)] + f(x, yz) = k(x,y,z) + f(x,y) + f(xy,z). \tag{16}$$

For each group G, we can construct a category, $\mathbf{Aut}_G$, whose objects are elements of the group of automorphisms $\mathrm{Aut}G$. For two elements α, β of $\mathrm{Aut}G$, we write

$$\mathrm{Hom}(\alpha, \beta) = \{c \in G | \alpha = \mu_c \circ \beta\}.$$

For two morphisms $c : \alpha \to \beta$; $d : \beta \to \gamma$ in $\mathbf{Aut}_G$, their composition is defined by $d \circ c = c + d$ (the sum in G).

The category $\mathbf{Aut}_G$ is a strict Gr-category with the tensor product defined by $\alpha \otimes \beta = \alpha \circ \beta$, and

$$(\alpha \xrightarrow{c} \alpha') \otimes (\beta \xrightarrow{d} \beta') = \alpha \otimes \beta \xrightarrow{c + \alpha' d} \alpha' \otimes \beta'. \tag{17}$$

The following proposition describes the reduced Gr-category of the Gr-category of an abstract kernel.

Proposition 13. *Let (Π, G, ψ) be an abstract kernel and $[k] \in H^3(\Pi, ZG)$ be its obstruction, $\mathbb{S} = (\Pi', C, h)$ be the reduced Gr-category of $\mathbf{Aut}_G$. Then*

i) $\Pi' = \pi_0(\mathbf{Aut}_G) = \mathrm{Aut}G/\mathrm{In}G$, $C = \pi_1(\mathbf{Aut}_G) = ZG$,

ii) *$\psi^* h$ belongs to the cohomology class of k.*

Proof. i) It follows from the definition of the category $\mathbf{Aut}_G$ and the reduced Gr-category.

ii) Let $(H, \widetilde{H})$ be a canonical Gr-equivalence from $\mathbb{S}$ to $\mathbf{Aut}_G$. Then, the following diagram

$$\begin{array}{ccccc} Hr(HsHt) & \xrightarrow{id \otimes \widetilde{H}} & HrH(st) & \xrightarrow{\widetilde{H}} & H(r(st)) \\ \| & & & & \downarrow{\scriptstyle H(\bullet, h(r,s,t))} \\ (HrHs)Ht & \xrightarrow{\widetilde{H} \otimes id} & H(rs)Ht & \xrightarrow{\widetilde{H}} & H((rs)t) \end{array} \tag{18}$$

commutes for all $r, s, t \in \Pi'$. Since $\mathbf{Aut}_G$ is a strict Gr-category, we have

$$\gamma_\alpha(u) = u,\ \forall \alpha \in \mathbf{Aut}_G,\ \forall u \in ZG = C.$$

By the definition of H, we obtain $H(\bullet, c) = c$, $\forall c \in C$. From the commutativity of the diagram (18) and the relation (17), we have

$$Hr[g(s,t)] + g(r, st) = g(r,s) + g(rs,t) - h(r,s,t), \tag{19}$$

where $g = g_H : \Pi' \times \Pi' \to G$ is associated to $\widetilde{H}$.

For the abstract kernel (Π, G, ψ), choose a function $\varphi = H.\psi : \Pi \to \mathrm{Aut}(G)$. Clearly, $\varphi(1) = id_G$. Moreover, since

$$\widetilde{H}_{\psi(x),\psi(y)} : H\psi(x)H\psi(y) \to H\psi(xy)$$

is a morphism in $\mathbf{Aut}_G$, for all $x, y \in \Pi$, we have

$$\varphi(x)\varphi(y) = H\psi(x)H\psi(y) = \mu_{f(x,y)}H\psi(xy) = \mu_{f(x,y)}\varphi(xy),$$

where $f(x,y) = \widetilde{H}_{\psi(x),\psi(y)}$. The pair (φ, f) satisfies the relation (15) and thus, it is a factor set of the abstract kernel (Π, G, ψ). It induces an *obstruction* $k(x,y,z) \in Z^3(\Pi, ZG)$ satisfying (16). Now, for $r = \psi(x),\ s = \psi(y),\ t = \psi(z)$, the equation (19) becomes

$$\varphi(x)[f(y,z)] + f(x,yz) = +f(x,y) + f(xy,z) - (\psi^*h)(x,y,z).$$

In comparison with (16), $[\psi^*h] = [k]$. □

We now use Proposition 13 and the theorem on the realization of the obstruction in the group extension problem to prove Proposition 15. First, we need the following lemma.

Lemma 14. *Let $\mathbb{H}$ be a strict Gr-category and $S_{\mathbb{H}} = (\Pi, C, h)$ be its reduced Gr-category. Then, for each group homomorphism $\psi : \Pi' \to \Pi$, there exists a strict Gr-category $\mathbb{G}$ which is Gr-equivalent to the Gr-category $\mathbb{J} = (\Pi', C, h')$, where C is regarded as a Π'-module with an operator $xc = \psi(x)c$, and h' belongs to the same cohomology class as ψ^*h.*

Proof. We construct a strict Gr-category $\mathbb{G}$ as follows

$$\mathrm{Ob}(\mathbb{G}) = \{(x, X)|\ x \in \Pi', X \in \psi(x)\},$$
$$\mathrm{Hom}((x, X), (x, Y)) = \{x\} \times \mathrm{Hom}_{\mathbb{H}}(X, Y).$$

The tensor products on objects and on morphisms of $\mathbb{G}$ are defined by

$$(x, X) \otimes (y, Y) = (xy, X \otimes Y),$$
$$(x, u) \otimes (y, v) = (xy, u \otimes v).$$

The unit object of $\mathbb{G}$ is $(1, I)$, where I is the unit object of $\mathbb{H}$. One can verify that $\mathbb{G}$ is a strict Gr-category. Moreover, we have isomorphisms

$$\begin{array}{ll} \lambda : \pi_0\mathbb{G} \to \Pi' & \qquad f : \pi_1\mathbb{G} \to \pi_1\mathbb{H} = C \\ [(x, X)] \mapsto x & \qquad (1, c) \mapsto c \end{array}$$

and a Gr-functor $(F, \widetilde{F}) : \mathbb{G} \to \mathbb{H}$ given by

$$F(x, X) = X, \quad F(x, u) = u, \quad \widetilde{F} = id.$$

Let $S_F = (\phi, \widetilde{\phi}) : S_{\mathbb{G}} \to S_{\mathbb{H}}$ be a Gr-functor induced by the functor $(F, \widetilde{F})$ between the reduced Gr-categories. Then, we have

$$\phi(x, X) = F_0(x, X) = [F(x, X)] = [X] = \psi(x),$$
$$\phi(1, u) = F_1(1, u) = \gamma_{F(1,I)} F(1, u) = \gamma_I(u) = u,$$

where u is a morphism in $\mathbb{G}$. This means $F_0 = \psi\lambda$ and $F_1 = f$, or S_F is a functor of type $(\psi\lambda, f)$.

Now let $h_{\mathbb{G}}$ be the associativity constraint of $S_{\mathbb{G}}$. By Theorem 6, the obstruction of the pair $(\psi\lambda, f)$ must vanish in $H^3(\pi_0\mathbb{G}, \pi_1\mathbb{H}) = H^3(\pi_0\mathbb{G}, C)$, i.e.,

$$(\psi\lambda)^* h = f_* h_{\mathbb{G}} + \delta\widetilde{\phi}.$$

Now, if we denote $h' = f_* h_{\mathbb{G}}$, then the pair $(J, \widetilde{J})$, where $J = (\lambda, f)$ and $\widetilde{J} = id$, is a Gr-functor from $S_{\mathbb{G}}$ to $\mathbb{J} = (\Pi', C, h')$. Thus, the composition

$$\mathbb{G} \xrightarrow{(G,\widetilde{G})} S_{\mathbb{G}} \xrightarrow{(J,\widetilde{J})} \mathbb{J}$$

is a Gr-equivalence from $\mathbb{G}$ to $\mathbb{J} = (\Pi', C, h')$.

Finally, we prove that h' belongs to the same cohomology class as $\psi^* h$. Let $K = (\lambda^{-1}, f^{-1}) : (\Pi', C, h') \to S_{\mathbb{G}}$, then K together with $\widetilde{K} = id$ is a Gr-functor, and the composition

$$(\phi, \widetilde{\phi}) \circ (K, \widetilde{K}) : (\Pi', C, h') \to S_{\mathbb{H}}$$

is a Gr-functor making the following diagram commute

$$\begin{array}{ccc} S_{\mathbb{G}} & \xrightarrow{\phi} & S_{\mathbb{H}} \\ & {\scriptstyle K}\nwarrow \quad \nearrow{\scriptstyle \phi\circ K} & \\ & \mathbb{J} = (\Pi', C, h'). & \end{array}$$

Clearly, $\phi \circ K$ is a Gr-functor of type (ψ, id) and therefore its obstruction vanishes. By (8), we have $\psi^* h - h' = \partial g$, i.e., $[h'] = [\psi^* h]$. □

Proposition 15. *Each Gr-category is Gr-equivalent to a strict one.*

Proof. Let $\mathbb{C}$ be a Gr-category whose reduced Gr-category is $S_{\mathbb{C}} = (\Pi', C, k)$. By the theorem on realization of the obstruction (Theorem 9.2 Section IV [10]), the realization of 3-cocycle $k \in H^3(\Pi', C)$ is the group G with the center $ZG = C$ and a group homomorphism $\psi : \Pi' \to \mathrm{Aut}G/\mathrm{In}G$ such that ψ induces a Π'-module structure on C and the obstruction of the abstract kernel (Π', G, ψ) is k. By Proposition 13, $S_{\mathbf{Aut}_G}$ =$(\mathrm{Aut}G/\mathrm{In}G, C, h)$ is the reduced Gr-category of the strict Gr-category $\mathbf{Aut}_G$, where $[\psi^* h] = [k]$.

Applying Lemma 14 to $\mathbb{H} = \mathbf{Aut}_G$ we see that the homomorphism $\psi : \Pi' \to \mathrm{Aut}G/\mathrm{In}G$ defines a strict Gr-category $\mathbb{G}$ which is Gr-equivalent to the strict Gr-category $\mathbb{J} = (\Pi', C, h')$. The Π'-module structures of C on $S_{\mathbb{C}}$ and on $\mathbb{J}$ coincide. Moreover, $[\psi^* h] = [h']$. It follows that $[h'] = [k]$. So there is a function $g : \Pi' \times \Pi' \to C$ such that $h' - k = \partial g$. Then, by Theorem 6,

$$(K, \widetilde{K}) = (id_{\Pi'}, id_C, g) : S_{\mathbb{C}} \to \mathbb{J}$$

is a Gr-equivalence. Therefore, $\mathbb{C}$ is equivalent to the strict Gr-category $\mathbb{G}$. □
Readers can see an another proof of Proposition 15 in [16].

4.2 Gr-functors and the group extension problem

In this section, we obtain Schreier Theory for group extensions thanks to Theorem 7. We denote by

$$\mathrm{Ext}(\Pi, G)$$

the set of equivalence classes of group extensions of G by Π, and state the following theorem.

Theorem 16. *Let G and Π be groups.*

i) *There is a canonical partition*

$$\mathrm{Ext}(\Pi, G) = \coprod_{\psi} \mathrm{Ext}(\Pi, G, \psi),$$

where, for each group homomorphism $\psi : \Pi \to \mathrm{Aut}G/\mathrm{In}G$, $\mathrm{Ext}(\Pi, G, \psi)$ is the set of equivalence classes of group extensions $E : G \to B \to \Pi$ of G by Π inducing ψ.

ii) *Each abstract kernel* (Π, G, ψ) *determines a (normalized) third-dimensional cohomology class* $\mathrm{Obs}(\Pi, G, \psi) \in H^3(\Pi, ZG)$ *(with respect to the* Π*-module structure on* ZG *obtained via* ψ*), called the obstruction of* (Π, G, ψ)*. The abstract kernel has extensions if and only if its obstruction vanishes. Then, there is a bijection*

$$\mathrm{Ext}(\Pi, G, \psi) \leftrightarrow H^2(\Pi, ZG).$$

Below, each factor set (φ, f) of a group extension can be lifted to a Gr-functor $F : \mathrm{Dis}\Pi \to \mathbf{Aut}_G$, where $\mathrm{Dis}\Pi$ is regarded as a Gr-category of the type $(\Pi, 0, 0)$, and therefore we can classify all group extensions by Gr-functors.

Denote by

$$\mathrm{Hom}_{(\psi,0)}[\mathrm{Dis}\Pi, \mathbf{Aut}_G]$$

the set of homotopy class of Gr-functors from $\mathrm{Dis}\Pi$ to $\mathbf{Aut}_G$ inducing the pair of homomorphisms $(\psi, 0)$, we have.

Theorem 17. *There exists a bijection*

$$\Delta : \mathrm{Hom}_{(\psi,0)}[\mathrm{Dis}\Pi, \mathbf{Aut}_G] \to \mathrm{Ext}(\Pi, G, \psi).$$

Proof. Step 1: The construction of the group extension E_F *of* G *by* Π *induced by Gr-functor* F.

Let $(F, \widetilde{F}) : \mathrm{Dis}\Pi \to \mathbf{Aut}_G$ be a Gr-functor. Then, $\widetilde{F}_{x,y} = f(x, y)$ is a function from Π^2 to G such that

$$Fx \circ Fy = \mu_{f(x,y)} \circ F(xy). \tag{20}$$

The compatibilities of $(F, \widetilde{F})$ with the associativity and unit constraints, respectively, imply

$$Fx[f(y, z)] + f(x, yz) = f(x, y) + f(xy, z), \tag{21}$$

$$f(x, 1) = f(1, y) = 0. \tag{22}$$

Set $B_F = \{(a, x) | a \in G, x \in \Pi\}$ and the operation

$$(a, x) + (b, y) = (a + Fx(b) + f(x, y), xy).$$

Then, B_F is an extension of G by Π,

$$E_F : \;\; 0 \to G \xrightarrow{i} B_F \xrightarrow{p} \Pi \to 1,$$

where $i(a) = (a, 1), p(a, x) = x$. The relations (20), (21) imply the associativity of the operation in B_F. By the relation (22), $(0, 1)$ is the zero, while $(b, x^{-1}) \in B_F$ is negative of the element (a, x), where b is an element such that $Fx(b) = -a - f(x, x^{-1})$.

The *associated* homomorphism $\psi : \Pi \to \mathrm{Aut}G/\mathrm{In}G$ is determined by $\psi(x) = [\mu_{(0,x)}]$. By a simple calculation, we have $\mu_{(0,x)}(a, 1) = (Fx(a), 1)$. Let G and its image iG be identical, we obtain $\psi(x) = [Fx]$.

Step 2: F and F' are homotopic if and only if E_F and $E_{F'}$ are equivalent.

Let $F, F' : \mathrm{Dis}\Pi \to \mathbf{Aut}_G$ be Gr-functors and $\alpha : F \to F'$ be a homotopy. Then, by the definition of Gr-morphism, the following diagram commutes

$$\begin{array}{ccc} Fx \otimes Fy & \xrightarrow{\widetilde{F}} & F(xy) \\ \Big\downarrow{\scriptstyle \alpha_x \otimes \alpha_y} & & \Big\downarrow{\scriptstyle \alpha_{xy}} \\ F'x \otimes F'y & \xrightarrow{\widetilde{F'}} & F'(xy). \end{array}$$

That is,

$$\widetilde{F}_{x,y} + \alpha_{xy} = \alpha_x \otimes \alpha_y + \widetilde{F}'_{x,y},$$

or

$$f(x, y) + \alpha_{xy} = \alpha_x + F'x(\alpha_y) + f'(x, y). \tag{23}$$

Now, we write

$$\begin{array}{rcl} \beta : B_F & \to & B_{F'} \\ (a, x) & \mapsto & (a + \alpha_x, x). \end{array}$$

Note that $Fx = \mu_{\alpha_x} \circ F'x$, and by (23) one can see that β is a homomorphism. Moreover, it is an isomorphism making the following diagram commute, i.e., E_F and $E_{F'}$ are equivalent,

$$\begin{array}{lccccccccc} E_F : & 0 & \to & G & \xrightarrow{i} & B_F & \xrightarrow{p} & \Pi & \longrightarrow & 1 \\ & & & \Big\downarrow{\scriptstyle id} & & \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle id} & & \\ E_{F'} : & 0 & \to & G & \xrightarrow{i'} & B_{F'} & \xrightarrow{p'} & \Pi & \longrightarrow & 1. \end{array}$$

The converse can be obtained as we see by retracing our steps.

Step 3: Δ is a surjection.

Suppose that the group extension

$$E : 0 \to G \xrightarrow{i} B \xrightarrow{p} \Pi \to 1$$

associates to the homomorphism $\psi : \Pi \to \mathrm{Aut}G/\mathrm{In}G$. Select a "representative" $u_x, x \in \Pi$, in B, that is $p(u_x) = x$. In particular, choose $u_1 = 0$. Then, the elements of B can be written uniquely as $a + u_x$, for $a \in G, x \in \Pi$, and

$$u_x + a = \mu_{u_x}(a) + u_x.$$

The sum $u_x + u_y$ must be in the same coset as u_{xy}, so there are unique elements $f(x,y) \in G$ such that

$$u_x + u_y = f(x,y) + u_{xy}.$$

The function f is a factor set of the extension E. It satisfies the relations

$$\mu_{u_x}[f(y,z)] + f(x,yz) = f(x,y) + f(xy,z), \quad x,y,z \in \Pi, \tag{24}$$

$$f(x,1) = f(1,y) = 0. \tag{25}$$

We define a Gr-functor $F = (F, \widetilde{F})$: $\mathrm{Dis}\Pi \to \mathbf{Aut}_G$ as follows: $Fx = \mu_{u_x}$, $\widetilde{F}_{x,y} = f(x,y)$.

Clearly, the relations (24), (25) show that $(F, \widetilde{F})$ is a monoidal functor between Gr-categories. □

We now prove Theorem 16.

Let (Π, G, ψ) be an abstract kernel. For each $x \in \Pi$, choose $\varphi(x) \in \psi(x)$ such that $\varphi(1) = id_G$. The family of $\varphi(x)$'s induces a function $f : \Pi^2 \to G$ satisfying the relation (15). The pair (φ, f) induces an obstruction $k \in Z^3(\Pi, ZG)$ by the relation (16). Write $F(x) = \varphi(x)$, we obtain a functor $\mathrm{Dis}\Pi \to \mathbf{Aut}_G$.

Let $\mathbb{S} = (\mathrm{Aut}G/\mathrm{In}G, ZG, h)$ be the reduced Gr-category of $\mathbf{Aut}_G$. Then F induces the pair of group homomorphisms $(\psi, 0) : (\Pi, 0) \to (\mathrm{Aut}G/\mathrm{In}G, ZG)$ and by the relation (8) an obstruction of the functor F is $\psi^* h$. By Proposition 13, $[\psi^* h] = [k]$, i.e., the obstruction of the abstract kernel (Π, G, ψ) and that of the functor F coincide. Then, by Theorem 6, (Π, G, ψ) has extensions if and only if its obstruction vanishes.

According to Theorem 7, there is a bijection

$$\mathrm{Hom}_{(\psi,0)}[\mathrm{Dis}\Pi, \mathbf{Aut}_G] \leftrightarrow H^2(\Pi, ZG),$$

since $\pi_0(\mathrm{Dis}\Pi) = \Pi, \pi_1(\mathbf{Aut}_G) = ZG$. Together with Theorem 17, one obtains

$$\mathrm{Ext}(\Pi, G, \psi) \leftrightarrow H^2(\Pi, ZG).$$

This completes the proof of Theorem 16 .

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