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FURTHER RESULTS ON THE NEUTRIX COMPOSITION OF THE DELTA FUNCTION

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Abstract

Let F be a distribution in $\mathcal{D}'$ and let f be a locally summable function. The composition $F(f(x))$ of F and f is said to exist and be equal to the distribution $h(x)$ if the neutrix limit of the sequence $\{F_n(f(x))\}$ is equal to $h(x)$, where $F_n(x) = F(x) * \delta_n(x)$ for $n = 1, 2, \dots$ and $\{\delta_n(x)\}$ is a certain regular sequence converging to the Dirac delta. It is proved that the neutrix composition $\delta^{(s)}[\ln^r(1 + x_+^{1/r})]$ exists and is given by

$$\sum_{k=0}^{s} \sum_{i=0}^{kr+r-1} \binom{kr+r-1}{i} \frac{(-1)^{(r+1)k+r+s+i-1} s!(i+1)^{sr+r-1}}{2(sr+r-1)!k!} \delta^{(k)}(x)$$

for $s = 0, 1, 2, \dots$ and $r = 1, 2, \dots$. Further results are also proved.

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1. Introduction

In the theory of distributions, many arguments show that no meaning can be generally given to expressions of the form $F(f(x))$, where F is a distribution and f is a locally summable function.

Using the concepts of a neutrix and neutrix limit due to van der Corput [1], the third author gave a general principle for the discarding of unwanted infinite quantities from asymptotic expansions. This has been exploited in the context of distributions, particularly in connection with the composition of distributions, see [2, 3]. With Fisher's definition, Koh and Li gave a meaning to δ^r and $(\delta')^r$ for $r = 2, 3, \ldots$, see [12], and the more general form $(\delta^{(s)}(x))^r$ was considered by Kou and Fisher in [13]. More recently the r-th powers of the Dirac function $\delta(x)$ and the Heaviside function $H(x)$ for negative integers have been defined in [14] and [15] respectively.

In the following, we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support, let $\mathcal{D}[a, b]$ be the space of infinitely differentiable functions with support contained in the interval $[a, b]$ and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$.

Now let $\rho(x)$ be a function in $\mathcal{D}$ having the following properties:
(i) $\rho(x) = 0$ for $|x| \geq 1$,
(ii) $\rho(x) \geq 0$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\displaystyle\int_{-1}^{1} \rho(x)\, dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. Further, if F is an arbitrary distribution in $\mathcal{D}'$ and $F_n(x) = F(x) * \delta_n(x) = \langle F(x-t), \varphi(t)\rangle$, then $\{F_n(x)\}$ is a regular sequence converging to $F(x)$.

If f is an infinitely differentiable function having a single simple zero at the point $x = x_0$, then the distribution $\delta^{(r)}(f(x))$ is defined by

$$\delta^{(r)}(f(x)) = \frac{1}{|f'(x_0)|}\left[\frac{1}{|f'(x)|}\frac{d}{dx}\right]^r \delta(x - x_0) \tag{1}$$

for $r = 0, 1, 2, \ldots$, see [11].

The third author generalized equation (1) in [2] as follows:

Definition 1. *Let f be an infinitely differentiable function. We say that the neutrix composition $\delta^{(r)}(f(x))$ exists and is equal to h on the open interval (a, b), with $-\infty < a < b < \infty$, if*

$$\operatorname*{N-lim}_{n\to\infty} \int_{-\infty}^{\infty} \delta_n^{(r)}(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle$$

for all φ *in* $\mathcal{D}[a,b]$, *where* N *is the neutrix, see [1], having domain* N' *the positive and range* N'' *the real numbers, with negligible functions which are finite linear sums of the functions*

$$n^\lambda \ln^{r-1} n, \quad \ln^r n: \quad \lambda > 0, \quad r = 1, 2, \dots$$

and all functions which converge to zero in the usual sense as n *tends to infinity.*

Note that taking the neutrix limit of a function $f(n)$ is equivalent to taking the usual limit of Hadamard's finite part of $f(n)$.

Definition 1 was later generalized with the following definition in [3] and was originally called the neutrix composition of distributions.

Definition 2. *Let* F *be a distribution in* $\mathcal{D}'$ *and let* f *be a locally summable function. We say that the neutrix composition* $F(f(x))$ *exists and is equal to* h *on the open interval* (a,b), *with* $-\infty < a < b < \infty$, *if*

$$\mathop{\mathrm{N-lim}}_{n\to\infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle$$

for all φ *in* $\mathcal{D}[a,b]$, *where* $F_n(x) = F(x) * \delta_n(x)$ *for* $n = 1, 2, \dots$. *In particular, we say that the composition* $F(f(x))$ *exists and is equal to* h *on the open interval* (a,b) *if*

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle$$

for all φ *in* $\mathcal{D}[a,b]$.

The following theorem was proved in [4].

Theorem 1. *The neutrix composition* $\delta^{(s)}(\operatorname{sgn} x|x|^\lambda)$ *exists and*

$$\delta^{(s)}(\operatorname{sgn} x|x|^\lambda) = 0$$

for $s = 0, 1, 2, \dots$ *and* $(s+1)\lambda = 1, 3, \dots$ *and*

$$\delta^{(s)}(\operatorname{sgn} x|x|^\lambda) = \frac{(-1)^{(s+1)(\lambda+1)} s!}{\lambda[(s+1)\lambda - 1]!}\delta^{((s+1)\lambda-1)}(x)$$

for $s = 0, 1, 2, \dots$ *and* $(s+1)\lambda = 2, 4, \dots$.

Next two theorems were proved in [5].

Theorem 2. *The compositions* $\delta^{(2s-1)}(\operatorname{sgn} x|x|^{1/s})$ *and* $\delta^{(s-1)}(|x|^{1/s})$ *exist and*

$$\delta^{(2s-1)}(\operatorname{sgn} x|x|^{1/s}) = \frac{(2s)!}{2}\delta'(x), \quad \delta^{(s-1)}(|x|^{1/s}) = (-1)^{s-1}\delta(x)$$

for $s = 1, 2, \dots$.

Theorem 3. *The neutrix composition $\delta^{(s)}[\ln(1+|x|^{1/r})]$ exists on the interval $(-1,1)$ and*

$$\delta^{(s)}[\ln(1+|x|^{1/r})] = \sum_{k=0}^{m}\sum_{i=0}^{kr+r-1}\binom{kr+r-1}{i}\frac{(-1)^{s+i-1}[1+(-1)^k]r(i+1)^s}{2(k!)}\delta^{(k)}(x) \quad (2)$$

for $r=2,3,\ldots$ and $s=0,1,2,\ldots$, where m denotes the largest integer less than or equal $(s+1)/r-1$.

In particular, the composition $\delta^{(s)}[\ln(1+|x|^{1/r})]$ exists and

$$\delta^{(s)}[\ln(1+|x|^{1/r})] = 0 \quad (3)$$

for $r=2,3,\ldots$ and $s=0,1,2,\ldots,r-2$, and

$$\delta^{(r-1)}[\ln(1+|x|^{1/r})] = (-1)^{r-1}r!\delta(x) \quad (4)$$

for $r=2,3,\ldots$.

2. Main Results

We now prove the following theorem.

Theorem 4. *The neutrix composition $\delta^{(s)}[\ln^r(1+x_+^{1/r})]$ exists and*

$$\delta^{(s)}[\ln^r(1+x_+^{1/r})] = \sum_{k=0}^{s}\sum_{i=0}^{kr+r-1}\binom{kr+r-1}{i}\frac{(-1)^{(r+1)k+r+s+i-1}s!(i+1)^{sr+r-1}}{2(sr+r-1)!k!}\delta^{(k)}(x) \quad (5)$$

for $s=0,1,2,\ldots$ and $r=1,2,\ldots$.

Proof. To prove equation (5), we will first of all evaluate

$$\mathrm{N}\text{-}\lim_{n\to\infty}\langle\delta_n^{(s)}[\ln^r(1+x_+^{1/r})],\varphi(x)\rangle,$$

for an arbitrary function $\varphi(x)$ in $\mathcal{D}[a,1]$, where $a<0$.

By Taylor's Theorem, we have

$$\varphi(x) = \sum_{k=0}^{s}\frac{\varphi^{(k)}(0)}{k!}x^k + \frac{x^{s+1}}{(s+1)!}\varphi^{(s+1)}(\xi x),$$

where $0 < \xi < 1$. Then if $\varphi(x)$ in $\mathcal{D}[a,1]$, we have

$$
\begin{aligned}
&\operatorname*{N-lim}_{n\to\infty}\langle \delta_n^{(s)}[\ln^r(1+x_+^{1/r})], \varphi(x)\rangle \\
&\qquad = \operatorname*{N-lim}_{n\to\infty}\sum_{k=0}^{s}\frac{\varphi^{(k)}(0)}{k!}\int_a^1 \delta_n^{(s)}[\ln^r(1+x_+^{1/r})]x^k\,dx \\
&\qquad + \operatorname*{N-lim}_{n\to\infty}\frac{1}{(s+1)!}\int_a^1 \delta_n^{(s)}[\ln^r(1+x_+^{1/r})]x^{s+1}\varphi^{(s+1)}(\xi x)\,dx. \qquad (6)
\end{aligned}
$$

For large enough n, we have

$$
\begin{aligned}
&\int_a^1 \delta_n^{(s)}[\ln^r(1+x_+^{1/r})]x^k\,dx \\
&= n^{s+1}\int_a^1 \rho^{(s)}[n\ln^r(1+x_+^{1/r})]x^k\,dx \\
&= n^{s+1}\int_a^0 \rho^{(s)}[n\ln^r(1+x_+^{1/r})x^k\,dx + n^{s+1}\int_0^1 \rho^{(s)}[n\ln^r(1+x_+^{1/r})]x^k\,dx \\
&= n^{s+1}\rho^{(s)}(0)\int_a^0 x^k\,dx + n^{s+1}\int_0^1 \rho^{(s)}[n\ln^r(1+x^{1/r})]x^k\,dx \\
&= -\frac{n^{s+1}a^{k+1}\rho^{(s)}(0)}{k+1} + n^{s+1}\int_0^1 \rho^{(s)}[n\ln^r(1+x^{1/r})]x^k\,dx \\
&= E_1 + E_2. \qquad (7)
\end{aligned}
$$

It follows immediately that

$$
\operatorname*{N-lim}_{n\to\infty} E_1 = 0. \qquad (8)
$$

Making the substitution $t = n\ln^r(1+x^{1/r})$, we have

$$
\begin{aligned}
E_2 &= n^{s+1}\int_0^1 \rho^{(s)}[n\ln^r(1+x^{1/r})]x^k\,dx \\
&= n^{s+1-1/r}\int_0^1 t^{1/r-1}\{\exp[(t/n)^{1/r}]-1\}^{kr+r-1}\exp[(t/n)^{1/r}]\rho^{(s)}(t)\,dt \\
&= n^{s+1-1/r}\sum_{i=0}^{kr+r-1}\binom{kr+r-1}{i}(-1)^{kr+r+i-1} \\
&\qquad\qquad \times\int_0^1 t^{1/r-1}\exp[(i+1)(t/n)^{1/r}]\rho^{(s)}(t)\,dt, \qquad (9)
\end{aligned}
$$

where

$$
\begin{aligned}
n^{s+1-1/r} &\int_0^1 t^{1/r-1} \exp[(i+1)(t/n)^{1/r}] \rho^{(s)}(t)\, dt \\
&= \sum_{j=0}^{sr+r-2} \int_0^1 \frac{(i+1)^j t^{(j+1)/r-1}}{j! n^{(j+1)/r-s-1}} \rho^{(s)}(t)\, dt \\
&\quad + \frac{1}{(sr+r-1)!} \int_0^1 (i+1)^{sr+r-1} t^s \rho^{(s)}(t) + O(n^{-1/r}).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\operatorname*{N-lim}_{n\to\infty} n^{s+1-1/r} &\int_0^1 t^{1/r-1} \exp[(i+1)(t/n)^{1/r}] \rho^{(s)}(t)\, dt \\
&= \frac{(-1)^s s! (i+1)^{sr+r-1}}{2(sr+r-1)!} \qquad (10)
\end{aligned}
$$

for $i = 0, 1, 2, \ldots, kr + r - 1$ and it now follows from equations (9) and (10) that

$$
\operatorname*{N-lim}_{n\to\infty} E_2 = \sum_{i=0}^{kr+r-1} \binom{kr+r-1}{i} \frac{(-1)^{kr+r+s+i-1} s! (i+1)^{sr+r-1}}{2(sr+r-1)!}. \qquad (11)
$$

Then using equations (7), (8) and (11), we see that

$$
\begin{aligned}
\operatorname*{N-lim}_{n\to\infty} &\int_a^1 \delta_n^{(s)}[\ln^r(1+x_+^{1/r})] x^k\, dx \\
&= \sum_{i=0}^{kr+r-1} \binom{kr+r-1}{i} \frac{(-1)^{kr+r+s+i-1} s! (i+1)^{sr+r-1}}{2(sr+r-1)!}, \qquad (12)
\end{aligned}
$$

for $k = 0, 1, 2, \ldots, s$.

When $k = s + 1$, we have

$$
\begin{aligned}
&\int_0^1 \left| \delta_n^{(s)}[\ln^r(1+x_+^{1/r})] x^{s+1} \right| dx \\
&\leq n^{s+1-1/r} \int_0^1 t^{1/r-1} \{\exp[(t/n)^{1/r}] - 1\}^{sr+2r-1} \exp[(t/n)^{1/r}] |\rho^{(s)}(t)|\, dt \\
&= n^{s+1-1/r} \int_0^1 t^{1/r-1} [(t/n)^{1/r} + O(n^{-2/r})]^{sr+2r-1} [1 + O(n^{-1/r})] |\rho^{(s)}(t)|\, dt \\
&= n^{s+1-1/r} \int_0^1 t^{1/r-1} [(t/n)^{s+2-1/r} + O(n^{-(s+2)})] |\rho^{(s)}(t)|\, dt \\
&= O(n^{-1}) \qquad (13)
\end{aligned}
$$

and so if ψ is an arbitrary function in $\mathcal{D}[a,1]$, we have

$$\lim_{n\to\infty}\int_0^1 \left|\delta_n^{(s)}[\ln^r(1+x_+^{1/r})]x^{s+1}\psi(x)\right| dx = 0. \tag{14}$$

It then follows from equations (6), (12) and (14) that

$$\begin{aligned}
&\mathrm{N}\!-\!\lim_{n\to\infty}\langle\delta_n^{(s)}[\ln^r(1+x_+^{1/r})],\varphi(x)\rangle \\
&= \sum_{k=0}^{s}\sum_{i=0}^{kr+r-1}\binom{kr+r-1}{i}\frac{(-1)^{kr+r+s+i-1}s!(i+1)^{sr+r-1}\varphi^{(k)}(0)}{2(sr+r-1)!k!} \\
&= \sum_{k=0}^{s}\sum_{i=0}^{kr+r-1}\binom{kr+r-1}{i} \\
&\qquad\times\frac{(-1)^{(r+1)k+r+s+i-1}s!(i+1)^{sr+r-1}}{2(sr+r-1)!k!}\langle\delta^{(k)}(x),\varphi(x)\rangle,
\end{aligned}$$

proving equation (5) on the interval $[a,1]$.

Since $\delta_n^{(s)}[\ln^r(1+x_+^{1/r})] = 0$ for $x > 0$, it follows that equation (5) holds for $x > a$ and since $a < 0$ is arbitrary, it follows that equation (5) holds on the real line, completing the proof of the theorem.

Theorem 5. *The neutrix composition $\delta^{(s)}[\ln^r(1+|x|^{1/r})]$ exists and*

$$\begin{aligned}
&\delta^{(s)}[\ln^r(1+|x|^{1/r})] \\
&= \sum_{k=0}^{s}\sum_{i=0}^{kr+r-1}\binom{kr+r-1}{i} \\
&\qquad\times\frac{(-1)^{r+s+k+i-1}[1+(-1)^k]s!(i+1)^{sr+r-1}}{2(sr+r-1)!k!}\delta^{(k)}(x),
\end{aligned} \tag{15}$$

for $s = 0,1,2,\ldots$ and $r = 1,2,\ldots$.

Proof. To prove equation (15), we now have to evaluate

$$\mathrm{N}\!-\!\lim_{n\to\infty}\langle\delta_n^{(s)}[\ln^r(1+|x|^{1/r})],\varphi(x)\rangle,$$

for an arbitrary function $\varphi(x)$ in $\mathcal{D}[-1,1]$. By Taylor's Theorem, we have

$$\varphi(x) = \sum_{k=0}^{s}\frac{\varphi^{(k)}(0)}{k!}x^k + \frac{x^{s+1}}{(s+1)!}\varphi^{(s+1)}(\xi x),$$

where $0 < \xi < 1$. Then if φ is in $\mathcal{D}[-1,1]$, we have

$$\begin{aligned}
&\mathop{\mathrm{N-lim}}_{n\to\infty}\langle \delta_n^{(s)}[\ln^r(1+|x|^{1/r})], \varphi(x)\rangle \\
&= \mathop{\mathrm{N-lim}}_{n\to\infty}\sum_{k=0}^{s}\frac{\varphi^{(k)}(0)}{k!}\int_{-1}^{1}\delta_n^{(s)}[\ln^r(1+x^{1/r})]x^k\,dx \\
&\quad + \mathop{\mathrm{N-lim}}_{n\to\infty}\frac{1}{(s+1)!}\int_{-1}^{1}\delta_n^{(s)}[\ln^r(1+|x|^{1/r}]x^{s+1}\varphi^{(s+1)}(\xi x)\,dx. \qquad (16)
\end{aligned}$$

Since

$$\begin{aligned}
\int_{-1}^{0}\delta_n^{(s)}[\ln^r(1+|x|^{1/r})]x^k\,dx &= (-1)^k\int_{0}^{1}\delta_n^{(s)}[\ln^r(1+|x|^{1/r})]x^k\,dx \\
&= (-1)^k n^{s+1}\int_0^1 \rho^{(s)}[n\ln^r(1+x^{1/r})]x^k\,dx, \quad (17)
\end{aligned}$$

it follows from equations (9) and (11) that

$$\begin{aligned}
&\mathop{\mathrm{N-lim}}_{n\to\infty}\int_{-1}^{1}\delta_n^{(s)}[\ln^r(1+|x|^{1/r})]x^k\,dx \\
&\qquad = \frac{[1+(-1)^k]}{2}\sum_{i=0}^{kr+r-1}\binom{kr+r-1}{i}\frac{(-1)^{r+s+i-1}s!(i+1)^{sr+r-1}}{(sr+r-1)!} \qquad (18)
\end{aligned}$$

for $k = 0, 1, 2, \ldots, s$.

When $k = s+1$, we have as in the proof of equation (14),

$$\lim_{n\to\infty}\int_{-1}^{1}\left|\delta_n^{(s)}[\ln^r(1+|x|^{1/r})]x^{s+1}\psi(x)\right|dx = 0, \qquad (19)$$

for an arbitrary continuous function ψ. It then follows from equations (16), (18) and (19) that

$$\begin{aligned}
&\mathop{\mathrm{N-lim}}_{n\to\infty}\langle \delta_n^{(s)}[\ln^r(1+x^{1/r})], \varphi(x)\rangle \\
&= \mathop{\mathrm{N-lim}}_{n\to\infty}\sum_{k=0}^{s}\frac{\varphi^{(k)}(0)}{k!}\int_{-1}^{1}\delta_n^{(s)}[\ln^r(1+|x|^{1/r})]x^k\,dx \\
&\quad + \lim_{n\to\infty}\frac{1}{(s+1)!}\int_{-1}^{1}\delta_n^{(s)}[\ln^r(1+|x|^{1/r})]x^{s+1}\varphi^{(s+1)}(\xi x)\,dx \\
&= \sum_{k=0}^{s}\sum_{i=0}^{kr+r-1}\binom{kr+r-1}{i}\frac{(-1)^{r+s+i-1}[1+(-1)^k]s!(i+1)^{sr+r-1}\varphi^{(k)}(0)}{2(sr+r-1)!k!} \\
&= \sum_{k=0}^{s}\sum_{i=0}^{kr+r-1}\binom{kr+r-1}{i}\frac{(-1)^{r+s+k+i-1}[1+(-1)^k]s!(i+1)^{sr+r-1}}{2(sr+r-1)!k!}\langle\delta^{(k)}(x), \varphi(x)\rangle,
\end{aligned}$$

proving equation (15) on the interval $[-1,1]$. However, it is clear that outside this interval, $\delta_n^{(s)}[\ln^r(1+|x|^{1/r})] = 0$, and so equation (15) is proved on the real line.

Theorem 6. *The neutrix composition* $\delta^{(r^s-1)}[\ln^{1/r}(1+|x|)]$ *exists and*

$$\delta^{(r^s-1)}[\ln^{1/r}(1+|x|)] = \sum_{k=0}^{r^{s-1}-1}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{r^s-i-1}[1+(-1)^k]r(r^s-1)!(i+1)^{r^{s-1}-1}}{2(r^{s-1}-1)!k!}\delta^{(k)}(x) \quad (20)$$

for $s = 1,2,\ldots$ *and* $r = 2,3,\ldots$.

Proof. This time we must evaluate

$$\operatorname*{N-lim}_{n\to\infty}\langle\delta_n^{(r^s-1)}[\ln^{1/r}(1+|x|)],\varphi(x)\rangle,$$

for an arbitrary function $\varphi(x)$ in $\mathcal{D}[-1,1]$.

By Taylor's Theorem, we have

$$\varphi(x) = \sum_{k=0}^{r^{s-1}-1}\frac{\varphi^{(k)}(0)}{k!}x^k + \frac{x^{r^{s-1}}}{(r^{s-1})!}\varphi^{(r^{s-1})}(\xi x),$$

where $0<\xi<1$. Then if $\varphi(x)$ in $\mathcal{D}[-1,1]$, we have

$$\begin{aligned}&\operatorname*{N-lim}_{n\to\infty}\langle\delta_n^{(r^s-1)}[\ln^{1/r}(1+|x|)],\varphi(x)\rangle\\ &= \operatorname*{N-lim}_{n\to\infty}\sum_{k=0}^{r^{s-1}-1}\frac{\varphi^{(k)}(0)}{k!}\int_{-1}^{1}\delta_n^{(r^s-1)}[\ln^{1/r}(1+|x|)]x^k\,dx\\ &\quad + \operatorname*{N-lim}_{n\to\infty}\frac{1}{(r^{s-1})!}\int_{-1}^{1}\delta_n^{(r^s-1)}[\ln^{1/r}(1+|x|)]x^{r^{s-1}}\varphi^{(r^{s-1})}(\xi x)\,dx. \qquad (21)\end{aligned}$$

For large enough n, we have

$$\begin{aligned}&\int_{-1}^{1}\delta_n^{(r^s-1)}[\ln^{1/r}(1+|x|)]x^k\,dx = n^{r^s}\int_{-1}^{1}\rho^{(r^s-1)}[n\ln^{1/r}(1+|x|)]x^k\,dx\\ &= n^{r^s}[1+(-1)^k]\int_0^1\rho^{(r^s-1)}[n\ln^{1/r}(1+x)]x^k\,dx. \qquad (22)\end{aligned}$$

Making the substitution $t = n\ln^{1/r}(1+x)$, we have

$$\begin{aligned}
&n^{r^s}\int_0^1 \rho^{(r^s-1)}[n\ln^{1/r}(1+x)]x^k\,dx \\
&= rn^{r^s-r}\int_0^1 t^{r-1}\{\exp[(t/n)^r]-1\}^k\exp[(t/n)^r]\rho^{(r^s-1)}(t)\,dt \\
&= rn^{r^s-r}\sum_{i=0}^k\binom{k}{i}(-1)^{k-i}\int_0^1 t^{r-1}\exp[(i+1)(t/n)^r]\rho^{(r^s-1)}(t)\,dt,
\end{aligned}$$

where

$$rn^{r^s-r}\int_0^1 t^{r-1}\exp[(i+1)(t/n)^r]\rho^{(r^s-1)}(t)\,dt = \sum_{j=0}^{\infty}\int_0^1\frac{r(i+1)^j t^{r(j+1)-1}}{j!n^{r(j+1)-r^s}}\rho^{(r^s-1)}(t)\,dt.$$

It follows that

$$\begin{aligned}
\mathrm{N}\!-\!\lim_{n\to\infty} rn^{r^s-r}&\int_0^1 t^{r-1}\exp[(i+1)(t/n)^r]\rho^{(r^s-1)}(t)\,dt \\
&= \int_0^1\frac{r(i+1)^{r^{s-1}-1}t^{r^s-1}}{(r^{s-1}-1)!}\rho^{(r^s-1)}(t)\,dt \\
&= \frac{(-1)^{r^s-1}r(r^s-1)!(i+1)^{r^{s-1}-1}}{2(r^{s-1}-1)!}
\end{aligned} \tag{23}$$

for $i = 0, 1, 2, \ldots, k$ and so

$$\mathrm{N}\!-\!\lim_{n\to\infty} n^{r^s}\int_0^1\rho^{(r^s-1)}[n\ln^{1/r}(1+x)]x^k\,dx = \sum_{i=0}^k\binom{k}{i}\frac{(-1)^{r^s+k-i-1}r(r^s-1)!(i+1)^{r^{s-1}-1}}{2(r^{s-1}-1)!}. \tag{24}$$

It now follows from equations (22) and (24) that

$$\mathrm{N}\!-\!\lim_{n\to\infty}\int_{-1}^1\delta_n^{(r^s-1)}[\ln^{1/r}(1+|x|)]x^k\,dx = [1+(-1)^k]\sum_{i=0}^k\binom{k}{i}\frac{(-1)^{r^s+k-i-1}r(r^s-1)!(i+1)^{r^{s-1}-1}}{2(r^{s-1}-1)!}, \tag{25}$$

for $k = 0, 1, 2, \ldots, r^{s-1}-1$.

When $k = r^{s-1}$, we have

$$\begin{aligned}
&\int_0^1 \left|\delta_n^{(r^s-1)}[\ln^{1/r}(1+x))]x^{r^{s-1}}\right| dx \\
&\qquad \le rn^{r^s-r}\int_0^1 t^{r-1}\{\exp[(t/n)^r]-1\}^{r^{s-1}}\exp[(t/n)^r]|\rho^{(r^s-1)}(t)|\,dt \\
&\qquad = rn^{r^s-r}\int_0^1 t^{r-1}[(t/n)^r+O(n^{-2r})]^{r^{s-1}}[1+O(n^{-r})]|\rho^{(r^s-1)}(t)|\,dt \\
&\qquad = rn^{r^s-r}\int_0^1 t^{r-1}[(t/n)^{r^s}+O(n^{-(r^s+r)})]|\rho^{(r^s-1)}(t)|\,dt \\
&\qquad = O(n^{-r})
\end{aligned}$$

and so if ψ is an arbitrary function in $\mathcal{D}[a,1]$, we have

$$\lim_{n\to\infty}\int_0^1 \left|\delta_n^{(r^s-1)}[\ln^{1/r}(1+x)]x^{r^{s-1}}\psi(x)\right| dx = 0.$$

Then if φ is an arbitrary function in $\mathcal{D}[-1,1]$, we have

$$\lim_{n\to\infty}\int_{-1}^1 \left|\delta_n^{(r^s-1)}[\ln^{1/r}(1+|x|)]x^{r^{s-1}}\varphi^{r^{s-1}}(\xi x)\right| dx = 0 \tag{26}$$

and it follows from equations (21), (25) and (26) that

$$\begin{aligned}
&\mathrm{N}\!-\!\lim_{n\to\infty}\langle\delta_n^{(r^s-1)}[\ln^{1/r}(1+|x|)]x^k,\varphi(x)\rangle \\
&= \sum_{k=0}^{r^{s-1}-1}[1+(-1)^k]\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{r^s+k-i-1}r(r^s-1)!(i+1)^{r^{s-1}-1}\varphi^{(k)}(0)}{2(r^{s-1}-1)!k!} \\
&= \sum_{k=0}^{r^{s-1}-1}[1+(-1)^k]\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{r^s-i-1}(r^s-1)!(i+1)^{r^{s-1}-1}}{2(r^{s-1}-1)!k!}\langle\delta^{(k)}(x),\varphi(x)\rangle,
\end{aligned}$$

proving equation (20) on the interval $[-1,1]$. However, it is clear that outside this interval $\delta_n^{(r^s-1)}[\ln^{1/r}(1+|x|)] = 0$, and so equation (20) is proved.

Finally we have

Theorem 7. *The neutrix composition* $\delta^{(r^s-1)}(\ln^{1/r}|1+x|)$ *exists and*

$$\delta^{(r^s-1)}(\ln^{1/r}|1+x|) = \sum_{k=0}^{r^{s-1}-1}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^i r(r^s-1)!(i+1)^{r^{s-1}-1}}{(r^{s-1}-1)!k!}\delta^{(k)}(x) \tag{27}$$

for $s = 1, 2, \ldots$ *and* $r = 1, 3, 5, \ldots$.

Proof. This time we must evaluate

$$\operatorname*{N-lim}_{n\to\infty}\langle \delta_n^{(r^s-1)}(\ln^{1/r}|1+x|), \varphi(x)\rangle,$$

for an arbitrary function $\varphi(x)$ in $\mathcal{D}[-1,1]$.

For large enough n, we have on making the substitution $t = n\ln^{1/r}(1+x)$,

$$\begin{aligned}
&\int_{-1}^{1} \delta_n^{(r^s-1)}(\ln^{1/r}|1+x|)x^k\,dx \\
&= \int_{-1}^{1} \delta_n^{(r^s-1)}[\ln^{1/r}(1+x)]x^k\,dx \\
&= rn^{r^s-r}\int_{-1}^{1} t^{r-1}\{\exp[(t/n)^r]-1\}^k \exp[(t/n)^r]\rho^{(r^s-1)}(t)\,dt \\
&= rn^{r^s-r}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\int_{-1}^{1} t^{r-1}\exp[(i+1)(t/n)^r]\rho^{(r^s-1)}(t)\,dt,
\end{aligned}$$

where

$$\begin{aligned}
rn^{r^s-r}&\int_{-1}^{1} t^{r-1}\exp[(i+1)(t/n)^r]\rho^{(r^s-1)}(t)\,dt \\
&= \sum_{j=0}^{\infty}\int_{-1}^{1}\frac{r(i+1)^j t^{r(j+1)-1}}{j!n^{r(j+1)-r^s}}\rho^{(r^s-1)}(t)\,dt.
\end{aligned}$$

Noting that r^s-1 is an even integer when r is an odd integer, and using equation (23), it follows that

$$\begin{aligned}
\operatorname*{N-lim}_{n\to\infty} rn^{r^s-r}&\int_{-1}^{1} t^{r-1}\exp[(i+1)(t/n)^r]\rho^{(r^s-1)}(t)\,dt \\
&= \int_{-1}^{1}\frac{r(i+1)^{r^{s-1}-1}t^{r^s-1}}{(r^{s-1}-1)!}\rho^{(r^s-1)}(t)\,dt = \frac{r(r^s-1)!(i+1)^{r^{s-1}-1}}{(r^{s-1}-1)!}
\end{aligned}$$

for $i = 0, 1, 2, \ldots, k$ and so

$$\begin{aligned}
\operatorname*{N-lim}_{n\to\infty} n^{r^s-r}&\int_{-1}^{1}\rho^{(r^s-1)}[n\ln^{1/r}(1+x)]x^k\,dx \\
&= \sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{k-i}(r^s-1)!(i+1)^{r^{s-1}-1}}{(r^{s-1}-1)!}.
\end{aligned}$$

Thus,

$$\underset{n\to\infty}{\text{N}-\lim}\int_{-1}^{1}\delta_n^{(r^s-1)}[\ln^{1/r}|1+x|]x^k\,dx$$
$$=\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{k-i}(r^s-1)!(i+1)^{r^{s-1}-1}}{(r^{s-1}-1)!},\qquad(28)$$

for $k=0,1,2,\ldots,r^{s-1}-1$.

When $k=r^{s-1}$, it follows that $\int_{-1}^{1}\left|\delta_n^{(r^s-1)}[\ln^{1/r}(|1+x|)]x^{r^{s-1}}\right|=O(n^{-r})$ and so if φ is an arbitrary function in $\mathcal{D}[-1,1]$, then

$$\lim_{n\to\infty}\int_{-1}^{1}\left|\delta_n^{(r^s-1)}[\ln^{1/r}(|1+x|)]x^{r^{s-1}}\varphi^{(r^{s-1})}(x)\right|dx=0.\qquad(29)$$

Now let φ be an arbitrary function in $\mathcal{D}[-1,1]$. By Taylor's Theorem, we have $\varphi(x)=\sum_{k=0}^{r^{s-1}-1}\frac{\varphi^{(k)}(0)}{k!}x^k+\frac{x^{r^{s-1}}}{(r^{s-1})!}\varphi^{(r^{s-1})}(\xi x)$, where $0<\xi<1$. Then if φ is in $\mathcal{D}[-1,1]$ and using equations (28) and (29), we have

$$\begin{aligned}
&\underset{n\to\infty}{\text{N}-\lim}\langle\delta_n^{(r^{s-1}-1)}(\ln^{1/r}|1+x|),\varphi(x)\rangle\\
&=\underset{n\to\infty}{\text{N}-\lim}\sum_{k=0}^{r^{s-1}-1}\frac{\varphi^{(k)}(0)}{k!}\int_{-1}^{1}\delta_n^{(r^s-1)}[\ln^{1/r}|1+x|]x^k\,dx\\
&\quad+\underset{n\to\infty}{\text{N}-\lim}\frac{1}{(r^{s-1})!}\int_{-1}^{1}\delta_n^{(r^s-1)}(\ln^{1/r}(|1+x|)x^{r^s}\varphi^{(r^{s-1})}(\xi x)\,dx\\
&=\sum_{k=0}^{r^{s-1}-1}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{k-i}r(r^s-1)!(i+1)^{r^{s-1}-1}\varphi^{(k)}(0)}{(r^{s-1}-1)!k!}\\
&=\sum_{k=0}^{r^{s-1}-1}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{i}(r^s-1)!(i+1)^{r^{s-1}-1}}{(r^{s-1}-1)!k!}\langle\delta^{(k)}(x),\varphi(x)\rangle,
\end{aligned}$$

proving equation (27) on $[-1,1]$. However, it is clear that outside this interval $\delta_n^{(r^s-1)}[\ln^{1/r}(|1+x|)]=0$, and so equation (27) is proved on the real line.

For further results on the neutrix composition of distributions, see [7], [8], [9] and [10].

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