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LINEAR FRACTIONAL TRANSFORMATIONS OF CONTINUED FRACTIONS WITH BOUNDED PARTIAL QUOTIENTS IN THE FIELD OF FORMAL SERIES

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Abstract

Let θ be an irrational element in the field of formal series. Using a modification of the 1997 technique due to Lagarias and Shallit in the real numbers case, it is shown that if the continued fraction expansion of θ has bounded partial quotients, so does its linear fractional transformation.

1. Introduction

Let α be an irrational real number whose simple continued fraction is $[b_0, b_1, b_2, ...]$. We say that α has bounded partial quotients if $\sup_{i\geq 1} b_i < \infty$. Lagarias and Shallit in [4] proved, using the so-called Lagrange constant through a result of

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Cusick and Mendès France in [2], that if α has bounded partial quotients, so does its linear fractional transformation. We show here that this is also the case in the field of formal series.

Let $\mathbf{F} := \mathbb{F}((x^{-1}))$ be the field of formal series over a field $\mathbb{F}$, equipped with the usual degree valuation $|\cdot|$, so normalized that $|P(x)| = 2^{\deg P(x)}$ $(P \in \mathbb{F}[x] \setminus \{0\})$. It is well-known, see e.g. [1, Chapter 1], that every element $\xi \in \mathbf{F} \setminus \{0\}$ can be uniquely written as

$$\xi := \sum_{n=r}^{\infty} w_n x^{-n},$$

where $r \in \mathbb{Z}$, $w_n \in \mathbb{F}$ $(n \geq r)$ and $w_r \neq 0$, so that $|\xi| = 2^{-r}$. Define the head part of ξ by

$$[\xi] = \begin{cases} \sum_{n=r}^{0} w_n x^{-n} & \text{if } r \leq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and the distance to the head part as

$$\|\xi\| := |\xi - [\xi]|.$$

In $\mathbf{F}$, there is a continued fraction algorithm similar to the case of real numbers which we briefly recall now; for details, see [6]. Each element $\xi \in \mathbf{F} \setminus \{0\}$ can be uniquely represented as a continued fraction of the form

$$\xi = b_0 + \frac{1}{b_1 +} \frac{1}{b_2 +} \cdots := [b_0, b_1, b_2, \ldots],$$

where $b_0 \in \mathbb{F}[x]$ and $b_i \in \mathbb{F}[x] \setminus \mathbb{F}$ $(i \geq 1)$ are called partial quotients. Such continued fraction of ξ is finite if and only if $\xi \in \mathbb{F}(x)$.

Let θ be an irrational in $\mathbf{F}$ whose infinite continued fraction expansion is

$$\theta = [a_0, \ a_1, \ a_2, \ldots].$$

Define the n^{th} complete quotient and the n^{th} convergent, respectively, of the continued fraction of θ as

$$\theta_n = [a_n, \ a_{n+1}, \ a_{n+2}, \ldots], \quad \frac{A_n}{B_n} = [a_0, \ a_1, \ a_2, \ldots, a_n].$$

The partial numerators, A_n, and partial denominators, B_n, satisfy the recursions

$$A_{-1} = 1, \ A_0 = a_0, \ A_{n+1} = a_{n+1} A_n + A_{n-1} \quad (n \geq 0)$$

and

$$B_{-1} = 0, \ B_0 = 1, \ B_{n+1} = a_{n+1} B_n + B_{n-1} \quad (n \geq 0).$$

Define

$$K(\theta) := \sup_{i \geq 1} |a_i|, \quad K_\infty(\theta) := \limsup_{i \geq 1} |a_i|.$$

We say that θ has **bounded partial quotients** if $K(\theta)$ is finite. Clearly, $K_\infty(\theta) \le K(\theta)$ and $K(\theta)$ is finite if and only if $K_\infty(\theta)$ is finite.

Our main result reads:

Theorem 1 *Let* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{F}[x])$*, the group of all invertible* 2×2 *matrices with entries from* $\mathbb{F}[x]$*. If the continued fraction of an irrational element* $\theta \in \mathbf{F}$ *has bounded partial quotients, then*

$$\frac{1}{|\det\ M|} K_\infty(\theta) \le K_\infty\left(\frac{a\theta+b}{c\theta+d}\right) \le |\det M|\, K_\infty(\theta), \tag{1}$$

$$K\left(\frac{a\theta+b}{c\theta+d}\right) \le \max\{|\det M|\, K(\theta),\ |c(c\theta+d)|\}. \tag{2}$$

2. Auxiliary results

The first lemma collects basic properties of continued fractions whose straightforward proof is omitted.

Lemma 2 *Let* $\theta = [a_0,\ a_1,\ a_2, \ldots]$ *be an irrational element in* $\mathbf{F}$, A_n/B_n *its* n^{th} *convergent and* θ_n *its* n^{th} *complete quotient. Let* $\zeta \in \mathbf{F}\setminus\{0\}$*. We have, for* $n \ge 0$,

(i) $|B_{n+1}| = |a_{n+1}B_n| > |B_n|\,,\ \ |\theta_n| = |a_n|$;

(ii) $A_nB_{n-1} - A_{n-1}B_n = (-1)^{n-1}$, so that $\gcd(A_n, B_n) = 1$;

(iii) $\theta - \frac{A_n}{B_n} = \frac{(-1)^n}{B_n(\theta_{n+1}B_n + B_{n-1})}$;

(iv) $\frac{\zeta A_n + A_{n-1}}{\zeta B_n + B_{n-1}} = [a_0, a_1, a_2, \ldots, a_n, \zeta]$;

(v) A_n *is the head part of* $B_n\theta$.

From Lemma 2 (v), we have $\|B_n\theta\| = |B_n\theta - A_n|$, and so Lemma 2 (i) and (iii) together yield

$$|B_n|\ \ \|B_n\theta\| = \frac{1}{|\theta_{n+1} + B_{n-1}/B_n|} = \frac{1}{|a_{n+1}|}. \tag{3}$$

The result in the next lemma is known as the best approximation property, cf. Theorem 7.13 in [5] for the real case.

Lemma 3 *Let* θ *be an irrational element in* $\mathbf{F}$ *and* A_n/B_n *its* n^{th} *convergent. If* $u, v(\ne 0) \in \mathbb{F}[x]$ *satisfy, for some* $n \ge 0$,

$$|v\theta - u| < |B_n\theta - A_n|, \tag{4}$$

then $|v| \ge |B_{n+1}|$.

Proof. Suppose that

$$|v| < |B_{n+1}|. \tag{5}$$

Consider the system of linear equations (in y, z)

$$yB_n + zB_{n+1} = v \tag{6}$$

$$yA_n + zA_{n+1} = u. \tag{7}$$

By Lemma 2 (*ii*), $\det \begin{pmatrix} B_n & B_{n+1} \\ A_n & A_{n+1} \end{pmatrix} = (-1)^n$, and so

$$\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} (-1)^n A_{n+1} & (-1)^{n+1} B_{n+1} \\ (-1)^{n+1} A_n & (-1)^n B_n \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix},$$

implying that y and z are in $\mathbb{F}[x]$.

We claim that neither y nor z is zero. If $y = 0$, then $0 \neq v = zB_{n+1}$, and so $|v| \geq |B_{n+1}|$, which contradicts (5). Assume then that $y \neq 0$. If $z = 0$, then $u = yA_n$ and $v = yB_n$. Since $|y| \geq 1$, we have $|v\theta - u| = |y(B_n\theta - A_n)| \geq |B_n\theta - A_n|$, contradicting (4).

Next we show that

$$|y(B_n\theta - A_n)| \neq |z(B_{n+1}\theta - A_{n+1})|. \tag{8}$$

Suppose $|y(B_n\theta - A_n)| = |z(B_{n+1}\theta - A_{n+1})|$. By Lemma 2 (*i*) and (*iii*), we have

$$|B_i\theta - A_i| = \frac{1}{|\theta_{i+1}B_i + B_{i-1}|} = \frac{1}{|B_{i+1}|} \quad (i \geq 0),$$

and so $|yB_{n+2}| = |zB_{n+1}|$. Since $|yB_n| < |yB_{n+2}|$, the ultrametric inequality and (6) yield $|zB_{n+1}| = |v|$ implying that $|B_{n+1}| \leq |v|$, contradicting (5). Thus, (8) holds.

Finally, consider $|v\theta - u| = |y(B_n\theta - A_n) + z(B_{n+1}\theta - A_{n+1})|$. Using (8), the ultrametric inequality and $y \in \mathbb{F}[x] \setminus \{0\}$, we have

$$|v\theta - u| = \max\{|y(B_n\theta - A_n)|,\ |z(B_{n+1}\theta - A_{n+1})|\} \geq |y(B_n\theta - A_n)| \geq |B_n\theta - A_n|,$$

which contradicts (4), and the lemma follows. □

For irrational $\theta \in \mathbf{F}$, define its **type** and its **Lagrange constant**, respectively, by

$$L(\theta) = \sup_{|B| \geq 1} \left(|B|\ \|B\theta\|\right)^{-1},\ L_\infty(\theta) = \limsup_{|B| \geq 1} \left(|B|\,\|B\theta\|\right)^{-1}.$$

To determine the type and Lagrange constant, it suffices to use the partial denominators as we show now.

Lemma 4 *We have*

$$L(\theta) = \sup_{i\geq 0} \left(|B_i| \, \|B_i\theta\|\right)^{-1}, \;\; L_\infty(\theta) = \limsup_{i\geq 0} \left(|B_i| \, \|B_i\theta\|\right)^{-1}. \tag{9}$$

Proof. Let $B \in \mathbb{F}[x] \setminus \{0\}$. Since the continued fraction of any irrational is infinite, there exists $m \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}$ such that $|B_m| \leq |B| < |B_{m+1}|$. By Lemma 3,

$$\frac{1}{|B| \; \|B\theta\|} \leq \frac{1}{|B| \; \|B_m\theta\|} \leq \frac{1}{|B_m| \; \|B_m\theta\|},$$

and the result follows. □

Corollary 5 *A) For irrational $\theta \in \mathbf{F}$, we have*

$$K(\theta) = L(\theta), \quad K_\infty(\theta) = L_\infty(\theta). \tag{10}$$

B) Let $\phi = [d_0, d_1, d_2, \ldots]$, $\gamma = [e_0, e_1, e_2, \ldots]$ be two irrational elements in $\mathbf{F}$. If there exist $s_1, s_2 \in \mathbb{N}_0$ such that $|d_{s_1+i}| = |e_{s_2+i}|$ $(i \geq 0)$, then

$$K_\infty(\phi) = K_\infty(\gamma), \quad L_\infty(\phi) = L_\infty(\gamma).$$

Proof. Part A) follows immediately from the definition of $K(\theta)$, $K_\infty(\theta)$, (3) and Lemma 4. Part B) follows from at once the definition of K_∞, Lemma 4 and (10). □

The next lemma is proved by modifying the proofs of Theorems 172 and 175 of [3] in the real to the formal series case.

Lemma 6 *Let $\theta = [a_0, a_1, a_2, \ldots]$ be an irrational element in $\boldsymbol{F}$ with $|\theta| > 1$, and let $\psi = \frac{a\theta+b}{c\theta+d}$, where $a, b, c, d \in \mathbb{F}[x]$ are such that $|ad - bc| = 1$.*

1) *If $|c| > |d| > 0$, then b/d and a/c are two consecutive convergents of the continued fraction of ψ.*

2) *If b/d and a/c are the $(n-1)^{th}$ and n^{th} convergents of the continued fraction of ψ, respectively, then the $(n+1)^{th}$ complete quotient is of the form $\delta\theta$ for some $\delta \in \mathbb{F}^* := \mathbb{F}\backslash\{0\}$.*

3) *If the continued fraction of ψ is $[c_0, c_1, c_2, \ldots]$, then there exist $k, n \in \mathbb{N}_0$ such that*

$$|a_{k+i}| = |c_{n+i}| \quad (i \geq 0).$$

Proof. We first prove parts 1) and 2) simultaneously. Denote the finite continued fraction expansion of a/c by $[c_0, \; c_1, \ldots, c_n]$ and let A_n/B_n be its n^{th} (last) convergent. Since $|ad - bc| = 1$, we have $\gcd(a, c) = 1 = \gcd(A_n, B_n)$. Thus,

$$|A_n d - B_n b| = |ad - bc| = 1 = |A_n B_{n-1} - A_{n-1} B_n|,$$

yielding $A_n d - B_n b = \delta'(A_n B_{n-1} - A_{n-1}B_n)$ for some $\delta' \in \mathbb{F}^*$, and so

$$A_n(d - \delta' B_{n-1}) = B_n(b - \delta' A_{n-1}). \tag{11}$$

Since $\gcd(A_n, B_n) = 1$, the relation (11) gives

$$B_n|(d - \delta' B_{n-1}). \tag{12}$$

From $|B_n| = |c| > |d| > 0$, and $|B_n| > |B_{n-1}| \geq 0$, we get $|d - \delta' B_{n-1}| < |B_n|$, which is consistent with (12) only when $d - \delta' B_{n-1} = 0$, i.e., when $d = \delta' B_{n-1},\ b = \delta' A_{n-1}$. Consequently, $\psi = \frac{A_n \delta\theta + A_{n-1}}{B_n \delta\theta + B_{n-1}}$ for some $\delta \in \mathbb{F}^*$, and so by Lemma 2 (iv),

$$\psi = [c_0, c_1, \ldots, c_n, \delta\theta].$$

If we develop $\delta\theta$ as a continued fraction, we obtain $\delta\theta = [c_{n+1}, c_{n+2}, \ldots]$, with $|c_{n+1}| > 1$. Hence, $\psi = [c_0, c_1, \ldots, c_n, c_{n+1}, c_{n+2}, \ldots]$.

To prove part 3), from Lemma 2 (iv), we have

$$\theta = [a_0, a_1, \ldots, a_{k-1}, \theta_k] \ = \ \frac{A_{k-1}\theta_k + A_{k-2}}{B_{k-1}\theta_k + B_{k-2}},$$

which implies

$$\psi = \frac{P\theta_k + R}{Q\theta_k + S},$$

where

$$P = aA_{k-1} + bB_{k-1},\ R = aA_{k-2} + bB_{k-2},\ Q = cA_{k-1} + dB_{k-1},\ S = cA_{k-2} + dB_{k-2}$$

are in $\mathbb{F}[x]$ with $|PS - QR| = |(ad - bc)(A_{k-1}B_{k-2} - A_{k-2}B_{k-1})| = 1$. From Lemma 2 ($iii$), we have $|\theta - \frac{A_i}{B_i}| = \frac{1}{|B_i(\theta_{i+1}B_i + B_{i-1})|} < \frac{1}{|B_i^2|}$ $(i \geq 0)$, and so

$$A_{k-1} = \theta B_{k-1} + \frac{\beta_1}{B_{k-1}},\ \ A_{k-2} = \theta B_{k-2} + \frac{\beta_2}{B_{k-2}},$$

where $|\beta_1| < 1,\ |\beta_2| < 1$. Thus,

$$Q = (c\theta + d)B_{k-1} + \frac{c\beta_1}{B_{k-1}},\ \ S = (c\theta + d)B_{k-2} + \frac{c\beta_2}{B_{k-2}}.$$

Since $c\theta + d \neq 0,\ |B_{k-1}| > |B_{k-2}| \to \infty\ (k \to \infty)$, we have $|Q| > |S| > 0$ for all large k. For such k, part 1) and part 2) ensure that there exists $\delta \in \mathbb{F}^*$ such that $\delta\theta_k = \psi_n$ for some n, i.e., $|a_{k+i}| \ = \ |c_{n+i}|$ $(i \geq 0)$. □

Lemma 6 and Corollary 5 B) immediately yield:

Lemma 7 *Let θ be an irrational element in* $\mathbf{F}$, $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{F}[x])$, *and* $M(\theta) := \frac{a\theta+b}{c\theta+d}$. *If* $|\det M| = 1$, *then*

$$L_\infty(M(\theta)) \ = \ L_\infty(\theta).$$

For transformation with non-unit determinant, we have weaker results.

Lemma 8 *Let θ be an irrational in $\boldsymbol{F}$; $h,\ d_1,\ d_3 \in \mathbb{F}[x] \setminus \{0\}$ and $d_2 \in \mathbb{F}[x]$. Then*

$$L_\infty(h\theta) \leq |h| L_\infty(\theta) \tag{13}$$

$$L_\infty\left(\frac{d_1\theta + d_2}{d_3}\right) \leq |d_1 d_3|\ L_\infty(\theta). \tag{14}$$

Proof. If θ has unbounded partial quotients, i.e., $L_\infty(\theta) = \infty$, both inequalities are trivial. Now assume θ has bounded partial quotients. For $h \in \mathbb{F}[x]\setminus\{0\},\ k \in \mathbb{N}_0$, clearly,

$$\sup_{\deg B \geq k} (|Bh|\ \|Bh\theta\|)^{-1} \leq \sup_{\deg B \geq k} (|B|\ \|B\theta\|)^{-1}$$

and

$$\limsup_{|B|\geq 1} (|Bh|\ \|Bh\theta\|)^{-1} \leq \limsup_{|B|\geq 1} (|B|\ \|B\theta\|)^{-1}.$$

Consequently,

$$\begin{aligned} L_\infty(h\theta) &= \limsup_{|B|\geq 1} (|B|\ \|Bh\theta\|)^{-1} = |h| \limsup_{|B|\geq 1} (|Bh|\ \|Bh\theta\|)^{-1} \\ &\leq |h| \limsup_{|B|\geq 1} (|B|\ \|B\theta\|)^{-1} = |h|\ L_\infty(\theta), \end{aligned}$$

which verifies (13).

To verify (14), from Corollary 5 B) and (13), we have

$$\begin{aligned} L_\infty\left(\frac{d_1\theta + d_2}{d_3}\right) &= L_\infty\left(\frac{d_3}{d_1\theta + d_2}\right) \leq |d_3| L_\infty\left(\frac{1}{d_1\theta + d_2}\right) \\ &= |d_3| L_\infty(d_1\theta + d_2) \leq |d_1||d_3| L_\infty(\theta). \end{aligned}$$

□

3. Proof of Theorem 1

By Corollary 5, it suffices to prove the two results for $L_\infty,\ L$ in place of $K_\infty,\ K$, respectively. Let $\psi := \frac{a\theta+b}{c\theta+d} = M(\theta)$.

We start by showing that there exists $M_2 \in GL_2(\mathbb{F}[x])$ such that

$$|\det M_2| = 1,\ M_2 M = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \in GL_2(\mathbb{F}[x]),\ |\alpha\gamma| = |\det M|.$$

Write $M_2 = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$. To fulfil the matrix equality, it is required that $Ga + Hc = 0$.

If $a = 0$, then $c \neq 0$ and so we must take $H = 0$. Now choose $F \in \mathbb{F}^*,\ G = 1/F$ and arbitrary $E \in \mathbb{F}[x]$ to fulfil all requirements.

If $c = 0$, then $a \neq 0$ and we must take $G = 0$. Now choose $E \in \mathbb{F}^*,\ H = 1/E$ and arbitrary $F \in \mathbb{F}[x]$ to fulfil all requirements.

If both $a \neq 0$ and $c \neq 0$, then take $G = \mathrm{lcm}(a, c)/a$ and $H = -\mathrm{lcm}(a, c)/c$. Since $\gcd(G, H) = 1$, there are $\mu, \nu \in \mathbb{F}[x]$ such that $\mu G + \nu H = 1$. Taking $E = \nu$ and $F = -\mu$, all the requirements are fulfilled.

Having obtained such M_2, we apply Lemma 7 to get

$$L_\infty(\psi) = L_\infty(M_2(\psi)) = L_\infty(M_2 M(\theta)) = L_\infty\left(\frac{\alpha\theta + \beta}{\gamma}\right),$$

and the second inequality of (1) now follows from the inequality (14) of Lemma 8.

To prove the first inequality of (1), we consider the adjoint matrix

$$M' := \mathrm{adj}\ (M) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

which has $M'M = (\det\ M)I_2$, and so

$$M'(\psi) = M'(M(\theta)) = M'M(\theta) = \theta.$$

Applying the second inequality of (1) to ψ, we have

$$L_\infty(\theta) = L_\infty(M'(\psi)) \leq |\det\ M'| L_\infty(\psi) = |\det\ M| L_\infty(\psi),$$

and the result follows.

We turn now to the second assertion of Theorem 1. For each $B \in \mathbb{F}[x] \setminus \{0\}$, let

$$x_B = |B|\, \|B\psi\| = |B| \left| B\left(\frac{a\theta + b}{c\theta + d}\right) - A \right| \qquad \left(A = \left[B\left(\frac{a\theta + b}{c\theta + d}\right)\right]\right).$$

If $c = 0$, then $|\det M| = |ad| \neq 0$ and so

$$|ad|\, x_B = |aB|\, |aB\theta - (dA - bB)| \geq |aB|\ \|aB\theta\| \geq 1/L(\theta),$$

yielding

$$L(\psi) = \sup_{|B| \geq 1} (|B|\, \|B\psi\|)^{-1} \leq |ad| L(\theta),$$

which is the first term in the right hand expression of (2).

If $c \neq 0$, then

$$|c\theta + d|\, x_B = |B|\, |(Ba - Ac)\theta - (Ad - Bb)|. \tag{15}$$

Since θ has bounded partial quotients, both $K(\theta)$ and $K_\infty(\theta)$ are finite. The result of the first part shows then that $K_\infty(\psi)$ is finite and so is $K(\psi)$. Corollary

5 in turn shows that $L(\psi)$ is finite. Thus, there is an infinite sequence of non-zero approximations

$$x_{B^{(i)}} = |B^{(i)}| \left\| B^{(i)}\psi \right\|$$

such that

$$L(\psi) - \frac{1}{2^i} \leq \frac{1}{x_{B^{(i)}}} \leq L(\psi) \qquad (i \geq 0). \tag{16}$$

By taking a suitable subsequence, we may reduce to the case where either all of the approximations have $B^{(i)}a - A^{(i)}c = 0$ or all of them have $B^{(i)}a - A^{(i)}c \neq 0$.

We first treat the subcase $B^{(i)}a - A^{(i)}c = 0$ for all $i \geq 0$. Since $ad - bc = \det M \neq 0$, we have $A^{(i)}d - B^{(i)}b \in \mathbb{F}[x] \setminus \{0\}$ and so (15) gives

$$|c\theta + d|\, x_{B^{(i)}} = |B^{(i)}|\, |A^{(i)}d - B^{(i)}b| \geq 1.$$

Consequently,

$$L(\psi) - \frac{1}{2^i} \leq \frac{1}{x_{B^{(i)}}} \leq |c\theta + d| \leq |c(c\theta + d)| \qquad (i \geq 0).$$

Letting $i \to \infty$, we get the second term in the right hand expression of (2).

Finally, consider the subcase that $B^{(i)}a - A^{(i)}c \neq 0$ for all $i \geq 0$. From (15), we have

$$\begin{aligned} |c\theta + d| \left| \tfrac{B^{(i)}a - A^{(i)}c}{B^{(i)}} \right| x_{B^{(i)}} &= |B^{(i)}a - A^{(i)}c| |(B^{(i)}a - A^{(i)}c)\theta - (A^{(i)}d - B^{(i)}b)| \\ &\geq |B^{(i)}a - A^{(i)}c| \left\| (B^{(i)}a - A^{(i)}c)\theta \right\| \geq \frac{1}{L(\theta)}. \end{aligned} \tag{17}$$

Using the first inequality in (16) and the inequality (17), we get

$$\begin{aligned} L(\psi) - \frac{1}{2^i} \leq \frac{1}{x_{B^{(i)}}} &\leq |c\theta + d| \left| \frac{B^{(i)}a - A^{(i)}c}{B^{(i)}} \right| L(\theta) \\ &= |c\theta + d| \frac{|c|}{|B^{(i)}|} \left| \frac{B^{(i)}a - A^{(i)}c}{c} \right| L(\theta). \end{aligned} \tag{18}$$

Using the strong triangle inequality, we have

$$\begin{aligned} \left| B^{(i)}\left(\frac{a}{c}\right) - A^{(i)} \right| &\leq \max\left\{ \left| B^{(i)}\left(\frac{a\theta + b}{c\theta + d}\right) - B^{(i)}\left(\frac{a}{c}\right) \right|,\ \left| B^{(i)}\left(\frac{a\theta + b}{c\theta + d}\right) - A^{(i)} \right| \right\} \\ &= \max\left\{ \frac{|B^{(i)}||\det\ (M)|}{|c(c\theta + d)|},\ \frac{x_{B^{(i)}}}{|B^{(i)}|} \right\}. \end{aligned} \tag{19}$$

Combining (18) and (19) gives

$$L(\psi) - \frac{1}{2^i} \leq L(\theta) \max\left\{ |\det M|,\ |c(c\theta + d)| \frac{x_{B^{(i)}}}{|(B^{(i)})|^2} \right\}.$$

Using the first inequality in (16), i.e., $x_{B^{(i)}} \leq \frac{1}{L(\psi)-1/2^i}$, we deduce that

$$L(\psi) - \frac{1}{2^i} \leq \max\left\{ |\det M| L(\theta),\ \frac{|c(c\theta + d)|}{|(B^{(i)})|^2} \cdot \frac{L(\theta)}{L(\psi) - 1/2^i} \right\}. \qquad (20)$$

If $L(\theta) \geq L(\psi)$, then the inequality (2) holds trivially, using the first term in the right hand expression. If $L(\theta) < L(\psi)$, then letting $i \to \infty$ in (20), the ratio $\frac{L(\theta)}{L(\psi)-1/2^i}$ becomes ≤ 1 in the limit, and (2) follows. □

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