East-West J. of Mathematics: Vol. 10, No 1 (2008) pp. 45-55

ON A NEW GENERALIZED BETA DISTRIBUTION

M.A. Pathan*, Mridula Garg and Jaya Agrawal****

**Department of Mathematics*
Aligarh Muslim University, Aligarh-202002, India
e-mail: mapathan@gmail.com

***Department of Mathematics*
University of Rajasthan, Jaipur–302004, India
e-mail:gargmridula@gmail.com; jaya_agrawal2005@yahoo.co.in

Abstract

In the present paper we introduce a generalized beta distribution with five parameters and derive expressions for its distribution function, characteristic function and the r^{th} moment. The other main findings are the probability density function (p.d.f.) of the product of two and 'n' generalized beta random variables and the distribution of the mixed product of the new generalized beta random variables with H-function random variables. A number of known and new special cases have also been mentioned.

1 Introduction

The beta distribution is a commonly used distribution and is frequently employed to model data. In reliability and life testing experiments, many times the data are modeled by finite range distributions. Looking into the applications of finite range distributions, in this paper we define a new distribution, which is a generalization of the well known beta distribution and study its properties.

Key words: Characteristic function, generalized beta distribution, Meijer's G-function, mixed product, probability density function.
2000 AMS Mathematics Subject Classification: 60E05, 44A10, 33C60.

A new generalized beta distribution

$$f(x) = \begin{cases} Cx^{\alpha-1}(1-x)^{\beta-1}(1-\sigma x)^{-\rho}\exp(-\eta x) & ; 0 \le x \le 1 \\ 0 & ; \text{elsewhere} \end{cases}, \tag{1.1}$$

$\alpha,\ \beta > 0,\ 0 \le \sigma < 1$, η and ρ are real and

$$C^{-1} = B(\alpha, \beta)\Phi_1(\alpha, \rho;\ \alpha+\beta;\ \sigma, -\eta), \tag{1.2}$$

$B(\alpha, \beta)$ is the well known beta function and $\Phi_1(\cdot)$ is Humbert's confluent hypergeometric function given in Srivastava and Manocha [12, p.58, Eq.(36)].

Particular cases

1. On taking $\eta = 0$ in p.d.f. (1.1), we get the following distribution

$$f(x) = \begin{cases} C_1x^{\alpha-1}(1-x)^{\beta-1}(1-\sigma x)^{-\rho}\exp(-\eta x) & ; 0 \le x \le 1 \\ 0 & ; \text{elsewhere} \end{cases}, \tag{1.3}$$

$\alpha,\ \beta > 0,\ 0 \le \sigma < 1$, ρ is real, where $C_1^{-1} = B(\alpha, \beta)\,{}_2F_1(\alpha, \rho; \alpha+\beta; \sigma)$ and ${}_2F_1(\cdot)$ is Gauss hypergeometric function given in Srivastava and Manocha [12, p.29].

2. On taking $\rho = 0$ or $\sigma = 0$ in p.d.f. (1.1), we obtain the following distribution

$$f(x) = \begin{cases} C_2x^{\alpha-1}(1-x)^{\beta-1}\exp(-\eta x) & ; 0 \le x \le 1, \\ 0 & ; \text{elsewhere} \end{cases}, \tag{1.4}$$

$\alpha,\ \beta > 0$, η is real, where $C_2^{-1} = B(\alpha, \beta)\,{}_1F_1(\alpha; \alpha+\beta; -\eta)$ and ${}_1F_1(\cdot)$ is confluent hypergeometric function given in Srivastava and Manocha [12, p.36].

3. On taking $\beta = 1$ in (1.1), we obtain the following p.d.f.

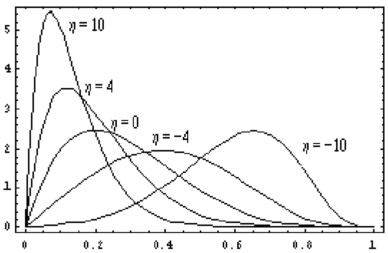
$$f(x) = \begin{cases} C_3x^{\alpha-1}(1-\sigma x)^{-\rho}\exp(-\eta x) & ; 0 \le x \le 1 \\ 0 & ; \text{elsewhere} \end{cases}, \tag{1.5}$$

$\alpha > 0,\ 0 \le \sigma < 1$, η and ρ are real and $C_3^{-1} = \dfrac{1}{\alpha}\,\Phi_1(\alpha, \rho; \alpha+1; \sigma, -\eta)$.

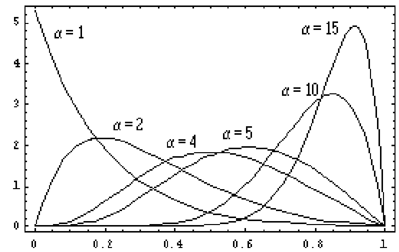
4. On taking $\beta = 1$ and $\eta = 0$ in (1.1), we obtain the following p.d.f.

$$f(x) = \begin{cases} C_4x^{\alpha-1}(1-\sigma x)^{-\rho} & ; 0 \le x \le 1 \\ 0 & ; \text{elsewhere} \end{cases}, \tag{1.6}$$

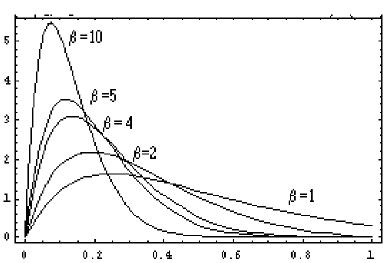
$\alpha > 0,\ 0 \le \sigma < 1$, ρ is real and $C_4^{-1} = \dfrac{1}{\alpha}\,{}_2F_1(\alpha, \rho; \alpha+1; \sigma)$.



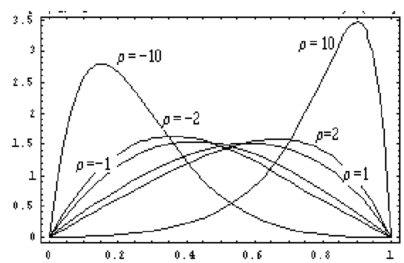
(a) Fig-1 α=2, β=5, σ=0, η=1



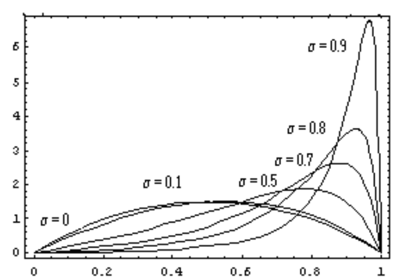
(b) Fig-2 β=2, σ=0, ρ=1, η=4



(c) Fig-3 α=1, σ=0, ρ=1, η=4



(d) Fig-4 α=2, β=2, σ=0.5, η=0



(e) Fig-5 α=2, β=2, ρ=4, η=0

Figure 1: Different shapes of the pdf (1.1) w.r.t. the change in parameters ρ, α, β, η, σ. illustrates the shape of p.d.f. (1.1) with different sets of values for the parameters α, β, ρ, η, σ. The effect of the parameters can clearly be seen.

5. On taking $\eta = 0 = \rho$ in (1.1), we get the well known beta distribution given in Johnson and Kotz [4, p.37]. On further taking $\alpha = 1 = \beta$ we get the uniform distribution given in Mathai [5].

Now we study some important properties of our p.d.f. $f(x)$ defined by (1.1).

The characteristic function

The characteristic function of p.d.f. $f(x)$ is given by

$$\Phi(t) \; = \; E(\exp(itx)) \; = \; \int_{-\infty}^{\infty} \exp(itx) f(x)\, dx \tag{1.7}$$

Substituting the value of $f(x)$ from (1.1) in (1.7) and using a known result given in Gradshteyn and Ryzhik [3, p.367, eq.(3.385)], we get

$$\Phi(t) \; = \; CB(\alpha,\beta)\Phi_1(\alpha,\rho;\, \alpha+\beta;\, \sigma, -(\eta - it)), \quad \alpha,\ \beta > 0,\ 0 \le \sigma < 1, \tag{1.8}$$

where $i = \sqrt{-1}$, $E(\cdot)$ stands for mathematical expectation and C is given by eq.(1.2).

The distribution function

The distribution function F(x) or the cumulative density function for the p.d.f. $f(x)$ is given by

$$F(x) \; = \; \int_{-\infty}^{\infty} f(x)\, dx$$

Substituting the value of $f(x)$ from (1.1) and evaluating the integral, we get

$$F(x) \; = \; Cx^{\alpha} F^{1:1;1;0}_{1:0;0;0} \left[\begin{array}{lll} (\alpha) & : (1-\beta);\ (\rho);\ \_\_ & ; \\ & & x, \sigma x, -\eta x \\ (\alpha+1) : & \_\_\_\_\_\_;\ \_\_\_;\ \_\_ & ; \end{array} \right], \tag{1.9}$$

$\alpha,\ \beta > 0,\ 0 \le \sigma < 1$, η and ρ are real. Here $F(\cdot)$ is the generalized Lauricella function given in the book Srivastava and Manocha [12, p.65] and C is given by (1.2).

The moments

The r^{th} moment of the p.d.f. $f(x)$ about the origin is given by

$$\mu'_r \; = \; E(x^r) \; = \; \int_{-\infty}^{\infty} x^r f(x)\, dx$$

Substituting the value of $f(x)$ from eq.(1.1) and evaluating the integral using a result given in Gradshteyn and Ryzhik [3, p.367, Eq.(3.385)], we get

$$\mu'_r CB(\alpha+r,\beta)\Phi_1(\alpha+\ r,\rho;\ r+\alpha+\beta;\ \sigma,-\eta), \tag{1.10}$$

$\alpha,\ \beta > 0,\ 0 \le \sigma < 1$, η and ρ are real and the constant C is given by (1.2).

2 Distribution of the product of independent generalized beta random variables

Theorem 1(a). Let X_1 and X_2 be two independent random variables (i.r.v.) following the generalized beta distribution given by (1.1), then the p.d.f. of the r.v. $Y = X_1X_2$ is given by

$$h(y) = C_1C_2\Gamma(\beta_1)\Gamma(\beta_2) \sum_{r_1,s_1,r_2,s_2=0}^{\infty} (\rho_1)_{r_1}(\rho_2)_{r_2}$$
$$\times G^{2,0}_{2,2}\left[y \;\middle|\; \begin{matrix} \alpha_1+\beta_1-1+r_1+s_1,\, \alpha_2+\beta_2-1+r_2+s_2 \\ \\ \alpha_1-1+r_1+s_1,\, \alpha_2-1+r_2+s_2 \end{matrix} \right]$$
$$\times \frac{(\sigma_1)^{r_1}}{r_1!}\frac{(-\eta_1)^{s_1}}{s_1!}\frac{(\sigma_2)^{r_2}}{r_2!}\frac{(-\eta_2)^{s_2}}{s_2!} \tag{2.1}$$

$\alpha_i,\ \beta_i > 0$, $0 \le \sigma_i < 1$, η_i and ρ_i are real

$$C_i^{-1} = B(\alpha_i,\beta_i)\,\Phi_1(\alpha_i,\rho_i;\,\alpha_i+\beta_i;\,\sigma_i,-\eta_i), \quad i=1,2 \tag{2.2}$$

Here $G(\cdot)$ stands for the Meijer's G-function given in Mathai and Saxena [9]. It is assumed that the series on the r.h.s. of (2.1) is convergent.

Proof. The p.d.f. of the product $Y = X_1X_2$ is given by Fox [2], as follows

$$h(y) = M^{-1}\Big\{M_s\{f_1(x_1)\}M_s\{f_2(x_2)\}\Big\} \tag{2.3}$$

$$= \int_{c-i\infty}^{c+i\infty} y^{-s}M_s\{f_1(x_1)\}M_s\{f_2(x_2)\}\,ds \tag{2.4}$$

where $M_s\{f_i(x_i)\}$ denotes the Mellin transform of $f_i(x_i)$ and $M^{-1}(\cdot)$ denotes the inverse Mellin transform. Using the definition (1.1) and a known result from Gradshteyn and Ryzhik [3, p.367, eq.(3.385.1)], we get

$$M_s\{f_i(x_i)\} = \int_0^1 x_i^{s-1}f_i(x_i)dx_i = C_iB(s+\alpha_i-1,\beta_i)$$
$$\times\Phi_1(s+\alpha_i-1,\rho_i;\,\alpha_i+\beta_i+s-1;\,\sigma_i,-\eta_i) \tag{2.5}$$

where C_i is given by (2.2).

We now substitute these values in (2.4), write the functions $\Phi_1(\cdot)$ in the series forms and interchange the order of summation and integration. Evaluating the inner integral thus obtained by using a known result from Gradshteyn and Ryzhik [3, p.687, eq.(6.422.19)], we arrive at the required result (2.1).

The above theorem can be generalized for the product of 'n' i.r.v. and the result can be stated in the following form:

Theorem 1(b). Let $X_1, X_2, \cdots, X_n$ be 'n' i.r.v. following the generalized beta distribution given by (1.1), then the p.d.f. of $Y = X_1 X_2 \cdots X_n$ is given by

$$h(y) = \prod_{i=1}^{n} C_i \Gamma(\beta_i) \sum_{r_i, s_i=0}^{\infty} (\rho_i)_{r_i} \frac{(\sigma_i)^{r_i} (-\eta_i)^{s_i}}{r_i!\ s_i!}$$

$$\times G_{n,n}^{n,0}\left[y \,\middle|\, \begin{matrix} \alpha_1+\beta_1-1+r_1+s_1,\ \alpha_2+\beta_2-1+r_2+s_2, \cdots, \alpha_n+\beta_n-1+r_n+s_n \\ \alpha_1-1+r_1+s_1,\ \alpha_2-1+r_2+s_2, \cdots, \alpha_n-1+r_n+s_n \end{matrix} \right] \tag{2.6}$$

$\alpha_i,\ \beta_i > 0$, $0 \le \sigma_i < 1$, η_i and ρ_i are real.

$$C_i^{-1} = B(\alpha_i, \beta_i)\Phi_1(\alpha_i, \rho_i;\ \alpha_i+\beta_i;\ \sigma_i, -\eta_i), \quad i = 1, \cdots, n.$$

In Theorem 1(b) if we specialize the generalized beta distribution to standard beta distribution, we obtain the result obtained by Springer and Thompson [10].

On reducing the generalized beta distribution occurring in Theorems 1(a) and 1(b) to the distributions as mentioned in Section 1, we can obtain the distributions of products of random variables having these probability density functions.

3 Distribution of the mixed product of independent random variables

We shall now obtain the distribution of the mixed product of i.r.v. X and Y when X follows the generalized beta distribution given by (1.1) and Y the H-function distribution given as follows

***H*-function distribution** (Mathai and Saxena [7])

$$g(x) = \begin{cases} Kx^{\lambda-1}\exp(-\gamma x)H_{P,Q}^{M,N}\left[ax^{\mu} \,\middle|\, \begin{matrix} (a_j,\alpha_j)_{1,P} \\ \\ (b_j,\beta_j)_{1,Q} \end{matrix} \right] & ; \quad x \geq 0 \\ 0 & ; \quad \text{elsewhere} \end{cases} \tag{3.1}$$

where

$$K^{-1} = \gamma^{-\lambda} H_{P+1,Q}^{M,N+1}\left[a\gamma^{-\mu} \,\middle|\, \begin{matrix} (1-\lambda,\mu)(a_j,\alpha_j)_{1,P} \\ \\ (b_j,\beta_j)_{1,Q} \end{matrix} \right] \tag{3.2}$$

Here $H_{P,Q}^{M,N}[ax^{\mu}]$ denotes the well known Fox H-function [1]. Throughout the paper it is assumed that this function always satisfies the conditions given in the books by Srivastava, Gupta and Goyal [11, p.11] and Mathai and Saxena [8].

Also

(i) $\mu > 0$, $\gamma > 0$

(ii) $\lambda + \mu \min\limits_{1\leq j\leq m}\left(\dfrac{b_j}{\beta_j}\right) > 0$

(iii) The parameters involved are so restricted that $f(x)$ remains non negative and

$$\int_0^{\infty} f(x)\,dx = 1 \tag{3.3}$$

The above distribution is very general in nature and generalizes many distributions such as generalized beta and gamma distribution, student-t distribution, normal distribution, exponential distribution, Cauchy, Rayleigh, Weibull, Maxwell distribution, generalized F-distribution, generalized hypergeometric distribution defined by Mathai and Saxena [6] and a distribution involving H-function defined by Srivastava and Singhal [13].

Theorem 2(a). The p.d.f. $h(z)$ of the product $Z = XY$ when X and Y are i.r.v. with p.d.f.s $f(x)$ and $g(y)$ given by (1.1) and (3.1) respectively is given as follows

$$h(z) = CK\Gamma(\beta)\sum_{t,s=0}^{\infty}(\rho)_t\,\frac{(\sigma)^t}{t!}\,\frac{(-\eta)^s}{s!}\,\gamma^{-\lambda+1}$$

$$\times \sum_{h=1}^{M} \sum_{r=0}^{\infty} \frac{\prod_{j=1}^{M} \Gamma(b_j - \beta_j \xi) \prod_{j=1}^{N} \Gamma(1 - a_j + \alpha_j \xi)}{\prod_{j=N+1}^{P} \Gamma(a_j - \alpha_j \xi) \prod_{j=M+1}^{Q} \Gamma(1 - b_j + \beta_j \xi)}$$

$$\times \frac{(-1)^r (a\gamma^{-\mu})^{\xi}}{(r)!\,\beta_h} G_{1,2}^{2,0}\left[\gamma z \;\middle|\; \begin{matrix} \alpha + \beta - 1 + t + s \\ \alpha - 1 + t + s,\ \lambda + \mu\xi - 1 \end{matrix} \right] \tag{3.4}$$

$\alpha,\ \beta > 0,\ 0 \le \sigma < 1,\ \eta$ and ρ are real, $\mu > 0,\ \lambda > 0,\ \lambda + \mu \min_{1 \le j \le m} \left(\frac{b_j}{\beta_j} \right) > 0$, $\xi = \frac{b_k + r}{\beta_k}$ and C and K are given by (1.2) and (3.2) respectively. It is assume that the series on the r.h.s. of (3.4) is convergent.

Proof. Following the lines of proof of Theorem 1(a) we can write the p.d.f. of the product $Z = XY$ as follows

$$h(z) = \int_{c-i\infty}^{c+i\infty} z^{-s} M_s\{f(x)\} M_s\{g(y)\} ds \tag{3.5}$$

The value of $M_s\{f(x)\}$ follows directly from the result given by eq.(2.5) and $M_s\{g(y)\}$ can be obtained using a known result from Srivastava, Gupta and Goyal [11, p.16, eq.(2.4.2)], as follows

$$M_s\{g(y)\} = \int_0^1 y^{s-1} g(y) dy = \gamma^{-\lambda+s+1} H_{P+1,Q}^{M,N+1}\left[a\gamma^{-\mu} \;\middle|\; \begin{matrix} (2-\lambda-s,\mu)(a_j,\alpha_j)_{1,P} \\ (b_j,\beta_j)_{1,Q} \end{matrix} \right] \tag{3.6}$$

$$\mu > 0, \quad \gamma > 0, \quad \lambda + \mu \min_{1 \le j \le M} \left(\frac{b_j}{\beta_j} \right) > 0,$$

Now, we substitute these values in (3.5), write the functions $\Phi_1(\cdot)$ and $H(\cdot)$ in their series forms (Srivastava, Gupta and Goyal [11, p.12]) and interchange the order of summation and integration. Evaluating the inner integral by using a known result from Gradshteyn and Ryzhik[3, p.687, eq.(6.422.19)] we arrive at the required result (3.4).

The above result can be generalized to give the p.d.f. $h(z)$ of the mixed product of 'n' generalized beta variables and '$m - n$' H-function random variables and the result can be stated in the following form:

Theorem 2(b). Let $X_1, X_2, \cdots, X_n$ be i.r.v. following the generalized beta distribution given by (1.1) and $Y_1 Y_2 \cdots Y_{m-n}$ be i.r.v. following the H-function distribution given by (3.1), then the p.d.f. of $Z = X_1X_2\cdots X_nY_1Y_2\cdots Y_{m-n}$ $(1 \le n < m)$ is given by

$$h(z) = \prod_{i=1}^{n} C_i\Gamma(\beta_i) \sum_{r_i,s_i=0}^{\infty} (\rho_i)_{r_i} \frac{(\sigma_i)^{r_i}}{r_i!} \frac{(-\eta_i)^{s_i}}{s_i!} \prod_{l=1}^{m-n} K_l \gamma_l^{-\lambda_l+1}$$

$$\times \sum_{h=1}^{M_l} \sum_{r'=0}^{\infty} \frac{\prod\limits_{\substack{j=1\\j\neq h}}^{M_l} \Gamma(b_j^{(l)} - \beta_j^{(l)}\xi_l) \prod\limits_{j=1}^{N_l} \Gamma(1 - a_j^{(l)} + \alpha_j^{(l)}\xi_l)}{\prod\limits_{j=N_l+1}^{P_l} \Gamma(a_j^{(l)} - \alpha_j^{(l)}\xi_l) \prod\limits_{j=M_l+1}^{Q_l} \Gamma(1 - b_j^{(l)} + \beta_j^{(l)}\xi^{(l)})} \frac{(-1)^{r^l}(a_l\gamma_l^{-\mu_l})^{\xi_l}}{(r^l)!\,\beta_h^{(l)}}$$

$$G_{n,m}^{m,0}\left[\gamma_l z \left| \begin{array}{l} \alpha_1+\beta_1-1+r_1+s_1,\ \alpha_2+\beta_2-1+r_2+s_2,\ \cdots,\alpha_n+\beta_n-1+r_n+s_n \\ \alpha_1-1+r_1+s_1,\cdots,\alpha_n-1+r_n+s_n,\ \lambda_1+\mu_1\xi_1-1,\ \cdots,\lambda_{m-n}+\mu_{m-n}\xi_{m-n}-1 \end{array}\right.\right] \tag{3.7}$$

$\alpha_i,\ \beta_i > 0$, $0 \le \sigma_i < 1$, η_i and ρ_i are real, $\mu_l > 0$, $\lambda_l > 0$, $\lambda_l + \mu_l \min\limits_{1\le j\le m}\left(\frac{b_j}{\beta_j}\right) > 0$,

where

$$C_i^{-1} = B(\alpha_i,\beta_i)\Phi_1(\alpha_i,\rho_i;\alpha_i+\beta_i;\ \sigma_i,-\eta_i), \qquad i = 1,2,\cdots,n$$

$\xi_l = \dfrac{b_h^{(l)} + r^l}{\beta_h^{(l)}}$ and

$$K_l^{-1} = \gamma_l^{-\lambda_l} H_{P_l+1,Q_l}^{M_l,N_l+1}\left[a_l\gamma_l^{-\mu_l}\left| \begin{array}{l} (1-\lambda_l,\mu_l)(a_j^{(l)},\alpha_j^{(l)})_{1,P_l} \\ (b_j^{(l)},\beta_j^{(l)})_{1,Q_l} \end{array}\right.\right],\ l = 1,2,\cdots,m-n.$$

It is assumed that the series on the r.h.s. of (3.7) is convergent.

In theorem 2(b) if we reduce the generalized beta distribution and the H-function distribution to standard beta and gamma distributions respectively, we obtain the p.d.f. of the product of 'n' beta and 'm-n' gamma random variables, as obtained by Springer and Thompson [10].

In theorem 2(a) and 2(b), we reduce the generalized beta distribution to the distribution as listed as listd in section I and the H-function distribution (3.1) to the distribution mentioned after equation (3.3), we can obtain their ditribution of the mixed product of independent variables having these p.d.f.'s which are defined on both finite and infinite ranges.

Acknowledgement

The authors are thankful to the anonymous referee for his useful suggestions which led to the present form of the paper.

References

[1] C. Fox, *The G and H-function as symmetrical Fourier kernels*, Trans. Amer. Math. Soc. **98**(1961), 395–429.

[2] C. Fox, *Some applications of Mellin transforms to the theory of bivariate statistical distributions*, Proc. Cambridge Philos. Soc. **53**(1957), 620–628.

[3] I.S. Gradshteyn and I.M. Rhyzik, *Table of Integrals, Series, and Products*, (fifth edition) Academic Press, New York, San Diego, 1994.

[4] N.L. Johnson and S. Kotz, *Continuous Univariate Distributions-2*, John Wiley and Sons, New York, 1970.

[5]] A.M. Mathai, *A Handbook of Generalized Special Functions for Statistical and Physical Sciences*, Clarendon Press, Oxford, 1993.

[6] A.M. Mathai and R.K. Saxena, *On a generalized hypergeometric distribution*, Metrika, **11** (1966), 127–132.

[7] A.M. Mathai and R.K. Saxena, *On the linear combinations of stochastic variables*, Metrika, **20**(3)(1973), 160–169.

[8] A.M. Mathai and R.K. Saxena, *The H-Function with Application in Statistical and Other Disciplines*, Wiley Eastern, New Delhi, 1978.

[9] A.M. Mathai and R.K. Saxena, *Generalized Hypergeometric Function with Applications in Statistics and Physical Sciences*, Springer-Verlag, Lecture Notes No.348, Heildelberg, 1973.

[10] M.D. Springer and W.E. Thompson, *The distribution of products of beta, gamma and Gaussian random variables*, SIAM J. Appl. Math. **18**(4)(1970), 721–737.

[11] H.M. Srivastava, K.C. Gupta and S.P. Goyal, *The H-Function of One and Two Variables with Applications*, South Asian Publishers, New Delhi and Madras, 1982.

[12] H.M. Srivastava and H.L. Manocha, *A Treatise on Generating Functions*, Ellis Horwood Ltd., Chichester; John Wiley and Sons, New York, 1984.

[13] H.M. Srivastava and J.P. Singhal, *On a class of generalized hypergeometric distributions*, Jñãnabha, Sect. A, **2**(1972), 969–977.