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INDEPENDENT SPACES GENERAYED BY A GRAPH

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Abstract

Using graph theory, this paper presents two ways to produce independence spaces. Simultaneously, it deals with some applications of the two new independence spaces.

1 Introduction and Preliminaries

J. Oxley points out in [4,p.73] that there is no single class of structures that one calls infinite matroids. There have been three main approaches to the study of infinite matroids, one is the independent set approach, another the closure operator approach, and the third approach is via lattices.

This paper is to use graph theory to deal with independence space which is one of the most fruitfully infinite matroids. It will concretely indicate how to produce independence spaces for a graph and use graph theory to obtain some properties of independence spaces.

The following in this section is devoted to a short summary of background knowledge we will need later on.

Definition 1 (1) Let $\{A, B, C, \cdots\}$ be a set of "points". If certain pairs of these points are connected by one or more "lines", the resulting configuration is called a *graph*[1,p.55]. Those points of $\{A, B, C, \cdots\}$ which are connected with at least one point are called *vertices* of the graph. The lines involved are called *edges* of the graph. An edge which connects A and B, i.e. whose endpoints are

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A and B, and which goes to A (and B), we shall designate by AB. If A is an endpoint of edge k, we shall say that A and k are *incident* to each other. If the set of vertices and the set of edges of a graph are both finite, the graph is called *finite*, otherwise *infinite*.

(2) If the vertices of the graph G' are at the same time vertices of G and the edges of G' are also edges of G, then G' is called a *subgraph* of G[1,p.57].

If all the edges of a graph can be listed in the form $AB, BC, CD, \cdots$, KL, LM, where each vertex and each edge can occur arbitrarily often, then the graph is characterized as a *walk* [1,p.61] . The walk is called *open* or *closed* depending on whether $A \neq M$, or $A = M$. If $A = M$, but $A, B, \cdots, L$ are distinct from one another, the closed walk is called a *cycle*.

A graph without a cycle is called *acyclic*[1, p.116]. If an acyclic graph is finite and connected(cf.[1,p.67]), then it is called a *tree*.

A graph is called *bipartite*[1, p.285] if each of its cycles contains an even number of edges.

An edge with identical ends is called a *loop*[2,p.3].

(3) For a given graph G, let π denote the set of vertices of G, and let k denote the set of edges. Let π' and k' have corresponding meanings for a graph G'. If there is a one-to-one correspondence between the sets π and π', on the one hand, and between k and k', on the other hand, in such a way that incident elements correspond to incident element, then graphs G and G' are said to be *isomorphic*[1,p.59].

Definition 2 ([3,pp.385-387& 4,p.74]) An *independence space* M is a set S together with a collection $\mathcal{I}$ of subsets of S (called *independent sets*) such that
(i1) $\mathcal{I} \neq \emptyset$;
(i2) A subset of an independent set is independent.
(i3) If I_1 and I_2 are finite members of $\mathcal{I}$ with $|I_1| < |I_2|$, then there exists x in $I_2 \setminus I_1$ such that $I_1 \cup x \in \mathcal{I}$;
(i4) If $X \subseteq S$ and every finite subset of X is in $\mathcal{I}$, then X is in $\mathcal{I}$.

A *basis* of M is a maximal independent set.

Lemma 1 ([1,p.285] and [4,pp.75-76])

(1) A graph is a bipartite graph iff its vertices can be divided into two classes in such a way that only vertices of different classes are joined by an edge.

(2) Suppose M is an independence space on S having $\mathcal{I}$ as its collection of independent sets. For $X \subseteq S$, let $\mathcal{I}|X$ be defined by $\mathcal{I}|X = \{Y \subseteq X : Y \in \mathcal{I}\}$. Clearly $\mathcal{I}|X$ is the collection of independent sets of an independence space $M|X$ on X.

Let $\mathcal{I} \cdot X = \{Y \subseteq X : Y \cup B \in \mathcal{I}\}$ where B is a basis of $M|(E \setminus X)$. Then $\mathcal{I} \cdot X$ is the set of independent sets of an independence space $M \cdot X$ on X.

Remark 1 (1) In this paper, $G = (V(G), E(G))$ is a graph and $V(G), E(G)$ always denotes its family of vertices, edges respectively. Except a special express, a graph in this paper always means an infinite one.

(2) Following [2] and [3], an acyclic graph is also called a *forest*. In addition, two edges in a graph are *parallel* if they have common endpoints and are not loops. A graph is *simple* if it has no loops or parallel edges.

An edge e of G is said to be *contracted* if it is deleted and its ends are identified, the resulting graph is denoted by $G \cdot e$. Let $T \subseteq X \subseteq E(G)$. Then write $X \cdot T$ for the resulted configuration obtained from $G[X]$ by contracting the edges of T. Write $G \cdot T$ for the graph obtained from G by contracting the edges of T.

Suppose that V' is a nonempty subset of $V(G)$. The subgraph of G whose vertex set is V' and whose edge set is the set of those edges of G that have both ends in V' is called the subgraph of G *induced* by V' and is denoted by $G[V']$; we say that $G[V']$ is an *induced subgraph* of G. Now suppose that E' is a nonempty subset of $E(G)$. The subgraph of G whose vertex set is the set of ends of edges in E' and whose edge is E' is called the subgraph of G *induced* by E' and is denoted by $G[E']$; it is an *edge-induced subgraph* of G.

(3) Following [2,p.70] and [3,p.104], a *matching* of a graph G is a set X of edges such that no two members of X have a common endpoint. A matching M *saturates* a vertex v and v is said to be M*-saturated*, if some edge of M is incident with v. If every vertex of $X \subseteq V(G)$ is M-saturated, then we say that X is saturated by M.

(4) The knowledge about tree of a graph is seen [1,chapter IV, V& 2]; connectivity of a graph is seen [1,§3 in chapter I, chapter II and III].

(5) Let $X \subseteq E(G)$ and $|X| < \infty$. Then obviously, $G[X]$ is a finite subgraph of G.

(6) Similar to the definition in [3] and [4,p.46], we give definitions as follows: let S be a set, T is a *transversal* of $(A_j : j \in J)$, $A_j \subseteq S$, if there is a bijection $\psi : J \to T$ such that $\psi(j) \in A_j$ for all j in J. If $X \subseteq S$, then X is a *partial transversal* of $(A_j : j \in J)$, if for some subset K of J, X is a transversal of $(A_j : j \in K)$.

2 Generated by edges

Here, we will know how to use the set of edges of a graph to produce an independence space. Besides, it is to obtain some properties about minors of the new independence space.

Lemma 2 *Let $T \subseteq X \subseteq E(G)$. Then*

(1) $X \cdot T = G[X] \cdot T$. Say, $G[X \cdot T] = G[X] \cdot T$.

(2) $X \cdot T$ contains a cycle C_T of $G[X \cdot T]$ if and only if X contains a cycle C of G, where $C_T = C \cdot T_C$ with $T_C \subseteq T$.

Especially, X is a cycle of G if and only if $X \cdot T$ is a cycle of $G[X \cdot T]$.

Proof By a routine verification.

Theorem 1 *Let G be a graph. $\mathcal{I} = \{X : X \subseteq E(G), X$ does not contain a cycle of $G\}$. Then $\mathcal{I}$ is the collection of independent sets of an independence space on $E(G)$. Denote this independence space by $M < G >$.*

Proof It is easy to see that $\mathcal{I}$ satisfies (i1) and (i2).

We prove that $\mathcal{I}$ satisfies (i3). If not, then there exist $A, B \in \mathcal{I}$ with $|A| < |B| < \infty$, such that for any $b \in B \setminus A, A \cup b$ contains a cycle. Since A does not contain a cycle of G, it follows that the edge-induced subgraph $G[A]$ of G is a forest. In light of remark 1, $G[A]$ is finite. Similarly, $G[B]$ is also a finite forest. Suppose all the components of $G[A]$ is $G_1, \cdots, G_t$, then by definition 1, G_j is a tree $(j = 1, \cdots, t)$.

Let A_j, V_j be the set of edges, vertices of G_j respectively $(j = 1, \cdots, t)$. By [1,p.120,theorem 9&2,p.25,theorem 2.2], we have $|A_j| = |V_j| - 1$. Evidently, $A_i \cap A_j = \emptyset, (i \neq j, i, j = 1, \cdots, t)$ and $\bigcup\limits_{j=1}^{t} A_j = A$.

According to the property of A, B, we know that for any $b \in B \setminus A, A \cup b$ contains a cycle. Let a_1, a_2 be the two endpoints of b, i.e. $b = a_1a_2$. We know that it has $a_1, a_2 \in V_j, j \in \{1, \cdots, t\}$. Put $B_j = \{x : x = uv, x \in B, u, v \in V_j\}$ $(j = 1, \cdots, t)$. It is easy to get $\bigcup\limits_{j=1}^{t} B_j = B$ and $B_i \cap B_j = \emptyset$ for $i \neq j, i, j = 1, \cdots, t$. Since $B_j \subseteq B, B_j$ and (i2) follows that B_j does not contain a cycle, besides, any endpoint of an edge of B_j belongs to V_j. Therefore, $|B_j| \leq |V_j| - 1 = |A_j|$, and so $|B| = \sum\limits_{j=1}^{t} |B_j| \leq \sum\limits_{j=1}^{t} |A_j| = |A|$, a contradiction.

Next we will prove the hold of (i4) for $\mathcal{I}$.

Let $X \subseteq E(G)$ satisfy that for any finite subset $Y, Y \in \mathcal{I}$ holds. Suppose X contains a cycle C, then $|X| \not< \infty$. Let $T = X \setminus C$. Then evidently, $X \cdot T \subseteq C$ and by lemma 2, $X \cdot T$ is a cycle of $G[X \cdot T]$, i.e, a cycle of $G[X] \cdot T$. Let $Z \subseteq X \cdot T, (X \cdot T) \setminus Z \neq \emptyset$ and $|(X \cdot T) \setminus Z| < \infty$, then by lemma 2, $(X \cdot T) \cdot Z$ is still a cycle of $G[(X \cdot T) \cdot Z]$.

Since $(X \cdot T) \cdot Z \subseteq X$ and $|X \setminus (T \cup Z)| < \infty$, by the supposition, $(X \cdot T) \cdot Z \in \mathcal{I}$, and so $(X \cdot T) \cdot Z$ is not a cycle of $G[(X \cdot T) \cdot Z]$, and hence by lemma 2, C is not a cycle of G, a contradiction. Say, X does not contain a cycle of G. Namely, (i4) holds for $\mathcal{I}$, and hence, $(E(G), \mathcal{I})$ is an independence space.

We now discuss some properties of minors of $M < G >$.

Property 1 *Let G be a graph and $T \subseteq E(G)$. Then*

(1) $M < G[T] >= M < G > |T$.

(2) $M < G \cdot (E(G) \setminus T) >= M < G > \cdot T$.

Proof It is straightforward from theorem 1, definition 1 and remark 1.

3 Generated by vertices

This section presents a way for producing an independence space by the set of vertices of a graph and discusses its application in transversal theory.

Lemma 3 *Let $X \subseteq Z \subseteq V(G)$. X is saturated by a matching of G if and only if X is saturated by a matching of $G[Z]$.*

Proof Routine verification.

Theorem 2 *Let $G = (V(G), E(G))$ be a graph, $m_p(G) = \{X : X \subseteq V(G), X$ is saturated by a matching of $G\}$ and $m(G) = \{X : X \subseteq V(G), |X| < \infty, X$ is saturated by a matching of $G\}$. Then $m_p(G)$ satisfies (i1)-(i3) and $m(G)$ is the family of independent sets of an independence space on $V(G)$.*

Proof (i1) and (i2) are evidently satisfied by $m_p(G)$.

Let $X, Y \in m_p(G)$, and $|X| < |Y| < \infty$. Put $Z = X \cup Y$. $S(G[Z])$ is a graph obtained as follows: the vertices is Z and all the loops of $G[Z]$ are deleted, besides, for $v_1, v_2 \in Z$, if there are more than one edges of $G[Z]$ joined the pair of vertices v_1 and v_2, then select one and only one edge among them to join v_1 and v_2 and denote this edge by v_1v_2, denoting all the edges gained in such a way by E_s. Therefore, $S(G[Z]) = (Z, E_s)$ is a finite simple subgraph of $G[Z]$ according to the finiteness of $G(Z)$ by remark 1 and the definition of simple graph. It is not difficult to know that for any $A \subseteq Z$, up to isomorphism of graphs, A is saturated by a matching of $G[Z]$ if and only if A is saturated by a matching of $S(G[Z])$. By the discussion in [5,p.47,theorem 1.6.2], we get that for $X, Y \in m_p(G)$, i.e. $X, Y \in m_p(S(G[Z]))$, there exists $y \in Y \setminus X$ such that $X \cup y$ is saturated by a matching of $S(G[Z])$, and so $X \cup y$ is saturated by a matching of $G[Z]$, further, saturated by a matching of G in view of lemma 3. Namely, (i3) holds for $m_p(G)$.

(i4) holds evidently for $m(G)$. Besides, $m(G) \subseteq m_p(G)$ and the above discussion show us that $m(G)$ is the family of independent sets of an independence space on $V(G)$.

We notice that for $A \subseteq V(G), m(G, A)$ is defined as $m(G, A) = \{X : X \subseteq A, |X| < \infty$, the vertices in X are saturated by a matching of $G\}$. Obviously, $m(G, A) = m(G)|A$. By lemma 1, $m(G, A)$ is the family of independent sets of an independence space.

Suppose $\mathbb{A}$ is the family $(A_1, A_2, \cdots)$ of subsets of a set S. Construct an associated bipartite graph $\Delta = \Delta(\mathbb{A})$ as follows: the vertex set of Δ is $S \cup \{A_i : i = 1, 2, \cdots\}$ and join $x_i \in S$ to A_j by an edge if and only if $x_i \in A_j$. The graph Δ is evidently a bipartite graph. Besides, $X \subseteq S$ is a partial transversal of $\mathbb{A}$ if and only if X is saturated by a matching of Δ. Considering with theorem 2, we easily have

Theorem 3 *If $\mathbb{A}$ is a family of subsets of a set S. Then the set of all the finite partial transversal sets of $\mathbb{A}$(i.e. X is a partial transversal set of $\mathbb{A}$ and*

$|X| < \infty$) *is the family of independents sets of an independence space on* S.

Remark 2 (1) Let G be a bipartite infinite graph(see Figure 1) with $V(G) = (X, Y), X = \{x_i : i = 1, 2, \cdots\} \cup \{x\}, Y = \{y_i : i = 1, 2, \cdots\}$ and $E(G) = \{x_1y_1, y_1x_2, x_2y_2, y_2x_3, \cdots, x_{i-1}y_{i-1}, y_{i-1}x_i, x_iy_i, y_ix_{i+1}, x_{i+1}y_{i+1}, \cdots\} \cup \{y_\infty x\}$. Then for any finite subset $A \subseteq V(G)$, by the definition of matching of graph and lemma 3, A can be saturated by a matching of G, but X can not. Hence $m_p(G)$ does not satisfy (i4), i.e. $(V(G), m_p(G))$ is not an independence space.

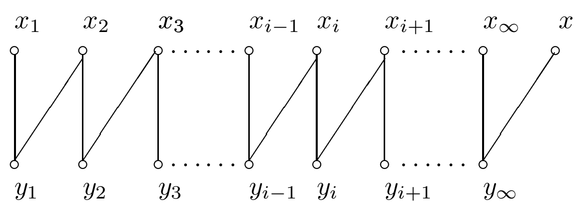

Figure 1

(2) In [3,p.385&4,p.74], a *pre-independence space* $M_p(S)$ is defined as a set S together with a collection $\mathcal{I}$(called *independent sets*) of subsets of S satisfying (i1)-(i3). Then by the proof of theorem 2, it follows that $(V(G), m_p(G))$ is a pre-independence space. For $A \subseteq V(G), m_p(G, A) = \{Z : Z \subseteq A$, the vertices in Z are saturated by a matching of $G\}$. Then $m_p(G, A) = m_p(G)|A$ is the collection of independent sets of a pre-independence space. Based on the discussion for $\mathbb{A}$, one has that the set of all the partial transversal sets of $\mathbb{A}$ is the collection of independent sets of a pre-independence space on S.

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