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DIGROUPS AND THEIR LINEAR PRESENTATIONS

Raúl Felipe

CIMAT, Callejón Jalisco S/N, Valenciana
Apdo. Postal 402, C.P. 36000
Guanajuato, Gto., México
ICIMAF, calle E No 309 esquina a 15
Vedado, La Habana. Cuba
e-mail: rfelipe ¡rfelipe@icmf.inf.cu¿

Abstract

In this paper we introduce the notion of digroup. It constitutes a version of the concept of group for sets with two products, such that, each element has an inverse which is the same for the two products. The basic results of the digroups are proved and some open problems are enunciated. Also, it is presented an approach of the representation theory for digroups where the notion of regular vector in Leibniz algebras, introduced recently by the author (see [3]), has a fundamental role.

1 Introduction

The Leibniz algebras and dialgebras first arose in algebraic K-theory and are objects of current interest. They were introduced by J.L.Loday and these constitute an extension of the concepts of Lie algebra and associative algebra respectively. More exactly, the Leibniz algebras are a generalization of Lie algebras, for which the skew-symmetry condition of the bracket is dropped and only the Jacobi identity is retained. On the other hand the definition of dialgebra is the following:

Definition 1. *A dialgebra is a vector space V together with two associative and bilinear operators, $\vdash$, $\dashv$, satisfying the following requirements*

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$$x \dashv (y \dashv z) = x \dashv (y \vdash z)\ ,$$
$$(x \vdash y) \dashv z = x \vdash (y \dashv z)\ ,$$
$$(x \dashv y) \vdash z = (x \vdash y) \vdash z,$$

for all x, y and z of V. These operators are called respectively, right and left products.

We say that e is a bar-unit of a dialgebra $(V, \vdash, \dashv)$ if $e \vdash v = v = v \dashv e$ for any $v \in V$. It is well known that if a dialgebra is given then it gives rise to a Leibniz algebra which is obtained by defining the bracket as

$$[x, y] = x \dashv y - y \vdash x$$

see [9] and [10] for more detail.

A basic problem in this context is the construction of dialgebras with bar-unit that are not associative algebras. Here it will be given an interesting dialgebra with a bar-unit: the so called "digroup dialgebra" which is not abelian. Thus we shall generalize the concept of group. For this we introduce the notion of digroup. Many questions arise around this concept that it should be considered in forthcoming papers. Some of them are:

i) Abstract harmonic analysis on topology digroup and construction of new dialgebras.

ii) Cohomology of digroups.

iii) To construct Hopf dialgebras and quantum digroups.

The aims of this paper are two: 1) to introduce the notion of digroup as a possible extensions of the group theory to sets with two products, 2) for these digroups to study their linear representations.

I would like to thank to M.Kinyon by correspondence and advices about this paper.

This paper is dedicated to the memory of my cousin Ana Margarita.

2 The notion of digroup.

We begin this section introducing three sets each one of which will be equipped with two suitable products. Subsets belong to these sets will be the first examples of digroups in our paper.

1) Let X be an arbitrary set. We define $x \vdash y = y$ for any x, $y \in X$ and $z \dashv w = z$ for all z, $w \in X$. Observe that in general $y = x \vdash y \neq x \dashv y = x$. Then

$$\begin{aligned}(x \vdash y) \vdash z &= y \vdash z = z,\\ (x \dashv y) \vdash z &= x \vdash z = z,\\ x \dashv (y \dashv z) &= x \dashv y = x,\\ x \dashv (y \vdash z) &= x \dashv z = x,\\ (x \vdash y) \dashv z &= y \dashv z = y,\\ x \vdash (y \dashv z) &= x \vdash y = y,\end{aligned}$$

Let us denote the set X equipped with these products by X_l.

2)Let V be a finite dimensional vector space over $\mathbb{C}$ and V^* its dual space. If φ is a nonzero element of V^* then we define the following products:

$$x \vdash y = \varphi(x)\, y, \qquad w \dashv z = \varphi(z)\, w$$

where x, y, z and w are elements of V. Now if x, y and z are elements of V we have

$$x \dashv (y \dashv z) = x \dashv (\varphi(z)\, y) = \varphi(\varphi(z)\, y)\, x = \varphi(z)\, \varphi(y)\, x,$$

on the other hand

$$x \dashv (y \vdash z) = x \dashv (\varphi(y)\, z) = \varphi(\varphi(y)\, z)\, x = \varphi(y)\, \varphi(z)\, x,$$

from the two last equations it now follows that $x \dashv (y \dashv z) = x \dashv (y \vdash z)$. Next, we would like to prove that $(x \vdash y) \dashv z = x \vdash (y \dashv z)$

$$(x \vdash y) \dashv z = (\varphi(x)\, y) \dashv z = \varphi(z)\, \varphi(x)\, y$$

and

$$x \vdash (y \dashv z) = x \vdash (\varphi(z)\, y) = \varphi(x)\, \varphi(z)\, y$$

then, as was claimed, the equality holds. Finally we have

$$(x \dashv y) \vdash z = (\varphi(y)\, x) \vdash z = \varphi(\varphi(y)\, x)\, z = \varphi(y)\, \varphi(x)\, z,$$

also we have

$$(x \vdash y) \vdash z = (\varphi(x)\, y) \vdash z = \varphi(\varphi(x)\, y)\, z = \varphi(x)\, \varphi(y)\, z,$$

so $(x \dashv y) \vdash z = (x \vdash y) \vdash z$. Let V_φ denote any vector space V equipped with these products.

3) Let V be a finite dimensional vector space and $\Pi : V \to V$ a linear operator such that $\Pi^2 = \Pi$ and

$$x \vdash y = \Pi x + y, \qquad w \dashv z = w + \Pi z$$

where x, y, z and w are elements in V. Then, it will be shown that $x \dashv (y \dashv z) = x \dashv (y \vdash z)$. In fact

$$\begin{aligned} x &\dashv (y \dashv z) = x \dashv (y + \Pi z) \\ &= x + \Pi (y + \Pi z) \\ &= x + \Pi y + \Pi^2 z = x + \Pi y + \Pi z, \end{aligned}$$

now

$$\begin{aligned} x &\dashv (y \vdash z) = x \dashv (\Pi y + z) \\ &= x + \Pi (\Pi y + z) \\ &= x + \Pi^2 y + \Pi z = x + \Pi y + \Pi z, \end{aligned}$$

this proves the equality required. On the other hand we have that $(x \vdash y) \dashv z = \Pi x + y + \Pi z$ and $x \vdash (y \dashv z) = \Pi x + y + \Pi z$, that is $(x \vdash y) \dashv z = x \vdash (y \dashv z)$. Finally, the last property of these products that we would like to show is $(x \dashv y) \vdash z = (x \vdash y) \vdash z$. Let us prove it

$$\begin{aligned} (x \dashv y) &\vdash z = (x + \Pi y) \vdash z \\ &= \Pi (x + \Pi y) + z \\ &= \Pi x + \Pi^2 y + z \\ &= \Pi x + \Pi y + z, \end{aligned}$$

and

$$\begin{aligned} (x \vdash y) &\vdash z = (\Pi x + y) \vdash z \\ &= \Pi (\Pi x + y) + z \\ &= \Pi^2 x + \Pi y + z \\ &= \Pi x + \Pi y + z, \end{aligned}$$

the last two equalities show that $(x \dashv y) \vdash z = (x \vdash y) \vdash z$ as required. Hereafter we adopt the notation $V(\Pi)$ for any vector space V with these products.

Historically the groups have been one of concepts more studied and it is one of the most fundamental concepts of contemporary mathematics, however after almost two century of development this theory has not been extended to sets with two special operators. An analysis of the sets X_l, V_φ and $V(\Pi)$ lead us to extend the group theory to sets with two particular products.

Definition 2. *A digroup is a pair (G, e) where G is a nonempty set and $e \in G$, such that, the set G is equipped with two associative maps called respectively right product and left product:*

$$\vdash : G \times G \to G,$$
$$\dashv : G \times G \to G,$$

satisfying the following requirements:

a)

$$x \dashv (y \dashv z) = x \dashv (y \vdash z)\,,$$
$$(x \vdash y) \dashv z = x \vdash (y \dashv z)\,,$$
$$(x \dashv y) \vdash z = (x \vdash y) \vdash z.$$

b) For any $g \in G$ it holds that

$$e \vdash g = g = g \dashv e,$$

the vector e is called the bar-unit of G.

c) For all $g \in G$ there exists an unique element $g^{-1} \in G$ such that with respect to e we have

$$g \vdash g^{-1} = e = g^{-1} \dashv g,$$

we say g^{-1} is the inverse of g.

It must be noted that from this definition it does not follow that e is the unique identity in G; in fact in general the digroup can have many identities (that is several vectors $\widetilde{e}$ such that $\widetilde{e} \vdash g = g = g \dashv \widetilde{e}$ for all $g \in G$). The notation (G, e) only suggests that between all the identities we have chosen e as the bar-unit of (G, e) with respect to which have means the point c) of the definition 2.

Remark 3. *After this paper was written the Prof. J.L.Loday has pointed to the author that when he discovered the Leibniz algebras in 1988 he was immediately led to the search of an analogue at the group level, because his motivation was algebraic K-theory. Later on he mentioned in several papers what kind of properties he expected for such a new algebraic structure which he called "coquecigrue" (which means roughly hypothetical bird in french). On the other hand we shall say that the notion of digroup discussed here has been also introduced independently by M.Kinyon and K.Liu, see* [2], [6] *and* [8] *for more detail.*

Remark 4. *It is clear that in the Definition* 2 *the important point is to decide about the analogue of the inverse: should one ask for one type of inverse or two types of inverse (one for each product)?, in this paper we have chosen the first option. We would like to thank the referee for telling me that in his book* [10] *Liu has considered a wider definition of digroup.*

Below we establish the basic properties of a digroup.

Example 5. *Let X be a set and x_0 an element in X. Then (X_l, x_0) is a digroup. In fact $x_0 \vdash z = z = z \dashv x_0$ for all $z \in X_l$, moreover $z \vdash x_0 = x_0 = x_0 \dashv z$ for any $z \in X_l$.*

Example 6. *The set* $V(\Pi)$ *is a digroup. Here* $e = \theta$ *and* $g^{-1} = -\Pi g$ *for all* $g \in G$.

Example 7. *Let* V *be a finite dimensional vector space and let* φ *be a nonzero element of* V^* *then the set* $\widehat{V_\varphi} = \{x \in V \mid \varphi(x) \neq 0\} \subset V_\varphi$ *is a digroup with bar-unit* $e = \frac{x_0}{\varphi(x_0)}$ *for some* $x_0 \in \widehat{V_\varphi}$. *If* $z \in \widehat{V_\varphi}$ *then* $z^{-1} = \frac{1}{\varphi(z)}e$. *These digroups will be called* φ*-digroups.*

Remark 8. *Let* (G, e) *be a digroup and* $f \in G$ *some unit of* G *then* $f^{-1} = e$.

Remark 9. *It is easy to show that if* (G, e) *is a digroup and* $g = e \dashv g$ *for any* $g \in G$ *then* $\vdash = \dashv$ *and* G *is a group, thus all groups are digroups.*

Remark 10. *M.Kinyon has seen in* [6] *that every digroup is a product of a group and a "trivial digroup", that is, a set which is both a left zero and right zero semigroup. For instance, example* 5 *of this paper is a "trivial digrou"; in example* 7 *the group in question is just a copy of the field and the trivial digroup is the kernel of the linear functional; in the example* 6*, the group is the range of* Π *and trivial digroup is the kernel of* Π.

3 Some elementary results of the digroups

A digroup (G, e) is called abelian if $x \dashv y = y \vdash x$ for all $x, y \in G$. A nonempty subset $H \subset G$ is said to be a subdigroup of (G, e), provided that (H, e) is a digroup for the same products that (G, e).

The proofs of all the results of this section were given in a different way in [2], when the author did not know the simultaneous works by Kinyon and Liu about digroups, however in the present paper we try to simplify these using the results in the references.

Lemma 11. *Let* (G, e) *be a digroup then for all* $g \in G$ *we have* $\left(g^{-1}\right)^{-1} = g \vdash e = e \dashv g$.

Proof. The proof follows from Lemma 4.3 (2) and Lemma 4.5 (1) in [6]. □

Theorem 12. *In order that* (H, e) *can be a subdigroup of a digroup* (G, e) *it is necessary and sufficient that for all* f, g, l, m, $n \in H$ *the elements* $f \vdash e$, $g^{-1} \vdash l$ *and* $m \dashv n^{-1}$ *belong to* H.

Proof. The conditions are clearly necessary. Let $x \in H$ then $\left(x^{-1} \vdash x\right) \vdash e = \left(x^{-1} \dashv x\right) \vdash e = e \in H$. Since now we know that e is an element of H, $x^{-1} \vdash e = x^{-1} \vdash \left(x \vdash x^{-1}\right) = \left(x^{-1} \dashv x\right) \vdash x^{-1} = e \vdash x^{-1} = x^{-1} \in H$, thus for all $x \in H$ also $x^{-1} \in H$. It follows from the Lemma 11 that $f \vdash g = (f \dashv e) \vdash g = (f \vdash e) \vdash g = \left(f^{-1}\right)^{-1} \vdash g \in H$ for all f, $g \in H$. Finally, also using the referred Lemma 11 we have $m \dashv l = m \dashv (e \vdash l) = m \dashv (e \dashv l) = m \dashv \left(l^{-1}\right)^{-1} \in H$ for any m, $l \in H$. Hence H is closed under the products $\vdash$ and $\dashv$. Consequently the theorem is proved. □

Observe that buried in the proof of this theorem is the result $x^{-1} \vdash e = x^{-1}$ for all $x \in G$ which leads to the fact obtained by Kinyon in [6] that the set of inverses in a digroup is a group (the group mentioned in the remark 10) in which $\vdash = \dashv$.

The intersection of two subdigroups (H, e) and (K, e) of a digroup (G, e) is not an empty set, since all subdigroups contain the element e. It is really a subdigroup of G. On the other hand, it is interesting to note that if (H_1, e), (H_2, e), $\cdots$, (H_n, e),$\cdots$ are subdigroups of a digroup (G, e) which form an ascending sequence, that is, $H_n \subset H_{n+1}$, $n = 1, 2, \cdots$, then $(\cup H_n, e)$ is a subdigroup of (G, e).

From now on all operators in any digroup will be denoted by $\vdash$ and $\dashv$, this should cause no confusion.

Definition 13. *A mapping γ of a digroup (G, e) into a digroup (G', e') is called a digroup-homomorphism (or homomorphism) if $\gamma(a \vdash b) = \gamma(a) \vdash \gamma(b)$ and also $\gamma(c \dashv d) = \gamma(c) \dashv \gamma(d)$ for all a, b, c, $d \in G$. A homomorphism one-to-one correspondence is called a digroup-isomorphism (or isomorphism).*

Let γ be a homomorphism of (G, e) into (G', e'), if $\gamma(G) = G'$, then $\gamma(e)$ is a unit of G'. Observe that $\gamma(e) \neq e'$ can happen. We now assume that $\gamma(e) = e'$, we shall show that $(\gamma(x))^{-1} = \gamma\left(x^{-1}\right)$ for all $x \in G$, in fact $e' = \gamma(e) = \gamma\left(x \vdash x^{-1}\right) = \gamma(x) \vdash \gamma\left(x^{-1}\right)$, on the other hand we have $e' = \gamma(e) = \gamma\left(x^{-1} \dashv x\right) = \gamma\left(x^{-1}\right) \dashv \gamma(x)$. Hence, $(\gamma(x))^{-1} = \gamma\left(x^{-1}\right)$.

Example 14. *If (G, e) and (G', e') are digroups, the direct product of G with G', denoted $G \times G'$ is the set of all ordered pairs (g, g') , where $g \in G$ and $g' \in G$, with the two operators $(g, g') \vdash (f, f') = (g \vdash f, g' \vdash f')$ and $(g, g') \dashv (f, f') = (g \dashv f, g' \dashv f')$. It is easy to check that $(G \times G', (e, e'))$ is a digroup containing homomorphic copies of G and G' namely, $G \times \{e'\}$ and $\{e\} \times G'$.*

Now we have

Theorem 15. *Let γ be a homomorphism of (G, e) into (G', e') such that $\gamma(e) = e'$. Then, if we define $N = \{g \in G \mid \gamma(g) = e'\}$, (N, e) is a subdigroup of (G, e) and is called the kernel of γ.*

Proof. Note that $e \in N$. Let us assume that $x \in N$ then $e' = \gamma\left(x^{-1} \dashv x\right) = \gamma\left(x^{-1}\right) \dashv e' = \gamma\left(x^{-1}\right)$. Thus, $x^{-1} \in N$. It is now obvious that if a, b, c, d and f are arbitrary elements of N then $a \vdash e$, $b^{-1} \vdash c$ and $d \dashv f^{-1}$ belong to N. The Theorem is proved. □

Theorem 16. *Let γ be a homomorphism of (G, e) into (G', e') such that $\gamma(e) = e'$. We define $I' = \{\gamma(g) \mid g \in G\} \subset G'$, then (I', e') is a subdigroup of (G', e').*

Proof. First let us note that $e' \in I'$. Suppose that $h' \in I'$ then $h' = \gamma(h)$ for some $h \in G$. Hence, we have $(h')^{-1} = (\gamma(h))^{-1} = \gamma\left(h^{-1}\right)$. It is then clear that $(h')^{-1} \in I'$. Finally, it is a simple matter to verify that if a', b', c', d' and f' are arbitrary elements of I' then $a' \vdash e'$, $(b')^{-1} \vdash c'$ and $d' \dashv (f')^{-1}$ are also elements of I'. □

A subdigroup (H, e) of the digroup (G, e) is called invariant or normal if $a^{-1} \vdash x \dashv a \in H$ for all $a \in G$ and any $x \in H$. Then we have

Proposition 17. *Under the conditions of the Theorem* 15*,* (N, e) *is an invariant or normal subdigroup.*

Proof. Let $z = \left(a^{-1} \vdash x\right) \dashv a$ where $x \in N$ and $a \in G$ then from Lemma 11 it follows that $\gamma(z) = \gamma\left(a^{-1} \vdash x\right) \dashv \gamma(a) = (\gamma(a))^{-1} \vdash e' \dashv \gamma(a) = e'$, this establishes that $z \in N$. Hence N is a normal subdigroup. □

Lemma 18. *Let* (G, e) *be a digroup and let* (H, e) *be a normal subdigroup of it. Then* $a^{-1} \vdash H \dashv a = H$ *for any* $a \in G$.

Proof. Since (H, e) is an invariant subdigroup we have that

$$a^{-1} \vdash H \dashv a \subset H \tag{1}$$

for all $a \in G$. Let $b \in G$ arbitrary then taking $a = b^{-1}$ in (1) we have $(b \vdash e) \vdash H \dashv b^{-1} \subset H$, now multiplying this inequality by b to the left and by b^{-1} to the right we obtain that

$$b^{-1} \vdash \left((b \vdash e) \vdash H \dashv b^{-1}\right) \dashv b \subset b^{-1} \vdash H \dashv b, \tag{2}$$

but $b^{-1} \vdash \left((b \vdash e) \vdash H \dashv b^{-1}\right) \dashv b = H$, hence from (2) it follows that

$$H \subset b^{-1} \vdash H \dashv b \tag{3}$$

for all $b \in G$. Thus (1) and (3) show that $c^{-1} \vdash H \dashv c = H$ for any $c \in G$. □

Corollary 19. *If* (H, e) *is a normal subdigroup of* (G, e)*, then also we have* $b \vdash H \dashv b^{-1} = H$ *for all* $b \in H$.

Proof. By the preceding Lemma $a^{-1} \vdash H \dashv a = H$ for any $a \in G$. Let $b \in G$ and $a = b^{-1}$ then we have $\left(b^{-1}\right)^{-1} \vdash H \dashv b^{-1} = H$ hence $(b \vdash e) \vdash H \dashv b^{-1} = H$, that is, $b \vdash (e \vdash H) \dashv b^{-1} = H$. □

Let (G, e) be a digroup, a homomorphism f: $(G, e) \to (G, e)$ is called an endomorphism of (G, e); an isomorphism f: $(G, e) \to (G, e)$ is called an automorphism of (G, e).

Proposition 20. *Let* (G, e) *be a digroup and the mapping* $u_a{:}G \to G$ *defined by the following form* $u_a g = a^{-1} \vdash g \dashv a$ *for* $a \in G$*. Then* u_a *is an automorphism of* (G, e)*. It is called an inner automorphism of* (G, e)*.*

Proof. If we define $a \circ b = a \vdash b \dashv a^{-1}$ for every a and b of G, then $u_a g = a^{-1} \circ g$ and we can use the Lemma 5.1 (2), (3) of [6], to see that $u_a(g \vdash f) = (u_a g) \vdash (u_a f)$ and $u_a(g \dashv f) = (u_a g) \dashv (u_a f)$. In order to prove that u_a is an automorphism of (G, e), we must show that u_a is one-to-one. Assume that $u_a f = u_a g$ then $a^{-1} \vdash f \dashv a = a^{-1} \vdash g \dashv a$, but it implies that $(a^{-1} \vdash f) = (a^{-1} \vdash g)$ and hence $f = g$. □

Theorem 21. *Let γ be an automorphism of the digroup (G, e), such that, $\gamma(e) = e$. Let $L(\gamma) = \{g \in G \mid \gamma(g) = g\}$ then $(L(\gamma), e)$ is a subdigroup of (G, e).*

Proof. It follows from Theorem 12. □

We would like to call the attention that the results found in the rest of the paper don't appear in [2].

4 Regular linear transformations

The representation theory of groups has nearly 110 years old and its importance have increased prominently ever since, for instance, at the present it has been an important source of inspiration for the mathematical physics. Next we describing a first approach for the representation theory of digroups. It is based on the formalism of regular vector in a dialgebra developed in [3].

In order to state our results we need the following definition introduced by the author in [3].

Definition 22. *An element x in a dialgebra $(\mathcal{U}, \vdash, \dashv)$ is said to be $(\vdash)-$regular $((\dashv)-$ regular$)$ with respect to a bar-unit e provided there exists $x_\vdash \in \mathcal{U}$ $(x_\dashv \in \mathcal{U},)$, such that $x \vdash x_\vdash = (e - x) + (x \vdash e)$ $(x_\dashv \dashv x = (e - x) + (e \dashv x))$. The element $x_\vdash$ $(x_\dashv)$ is called a $(\vdash)$-inverse $((\dashv)$-inverse$)$ for x with respect to e. An element which is both $(\vdash)-$regular and $(\dashv)-$regular with respect to e, is called regular if it has a $(\vdash)-$inverse that is also a $(\dashv)-$inverse, both with respect to e (that is $x_\vdash = x_\dashv$).*

It is interesting to note that if $\vdash$ is equal to $\dashv$ then these definitions coincide with the usual ones.

Let V a vector space of dimension n on $\mathbb{C}$. We have proved in the previous mentioned paper [3] that related with a fixed base $\{a_1, \cdots a_n\}$ of V is defined a structure of dialgebra on this vector space. We present this result here for further completeness in the writing of the paper (see [3] for more details). With the help of the base $\{a_1, \cdots a_n\}$ is introduced the usual inner product $\langle x, y\rangle = x_1\overline{y_1} + \cdots + x_n\overline{y_n}$ where $x = x_1a_1 + \ldots x_na_n$ and $y = y_1a_1 + \ldots y_na_n$. Then, we choose $e_0 \in V$ such that $\|e_0\|^2 = 1$ and define two products as following: $x \vdash y = \langle x, e_0\rangle y$, $z \dashv w = \langle w, e_0\rangle z$ for any x, y, z, $w \in V$. As was proved in [3] , $(V, \vdash, \dashv)$ is a dialgebra. It is called the principal dialgebra generated by V and the vector e_0 and from now it will be denoted by $V(e_0)$. It is possible to show that e_0 is a bar-unit of $V(e_0)$, that is, $e_0 \vdash x = x \dashv e_0$ for all $x \in V(e_0)$.

Let $L(V(e_0))$ be the set of all linear operators of V into itself.

Definition 23. *We say that $A \in L(V(e_0))$ is a regular linear transformation with respect to e_0 if Ae_0 is a regular vector of $V(e_0)$.*

It is not difficult to prove that Ae_0 is a regular vector of $V(e_0)$ if and only if $\langle Ae_0, e_0\rangle \neq 0$. Since Ae_0 is a regular vector in $V(e_0)$ then by the Definition 23, there exits a vector $r(e_0)$ such that

$$Ae_0 \vdash r(e_0) = (e_0 - Ae_0) + (Ae_0 \vdash e_0), \tag{4}$$

and also

$$r(e_0) \dashv Ae_0 = (e_0 - Ae_0) + (e_0 \dashv Ae_0), \tag{5}$$

hence $r(e_0) = \frac{(e_0 - Ae_0)}{\langle Ae_0, e_0\rangle} + e_0$.

It is easy to show that for any $x \in V(e_0)$ we have $(Ae_0 \vdash r(e_0)) \vdash x = x = x \dashv (r(e_0) \dashv Ae_0)$. In fact, let x be an arbitrary element of $V(e_0)$ then we have

$$\begin{aligned} (Ae_0 \vdash r(e_0)) \vdash x &= ((e_0 - Ae_0) + (Ae_0 \vdash e_0)) \vdash x \\ &= ((e_0 - Ae_0) \vdash x) + ((Ae_0 \vdash e_0) \vdash x) \\ &= x - (Ae_0 \vdash x) + (Ae_0 \vdash (e_0 \vdash x)) \\ &= x - (Ae_0 \vdash x) + (Ae_0 \vdash x) \\ &= x, \end{aligned}$$

of the same way it is proved that $x \dashv (r(e_0) \dashv Ae_0) = x$. Hence, we can define an operator A^{-1} in the following form $A^{-1}x = r(e_0) \vdash x = \langle r(e_0), e_0\rangle x = x \dashv r(e_0)$. Clearly the operator A^{-1} is linear and moreover $A \vdash A^{-1} = I = A^{-1} \dashv A$.

The collection of all the regular linear transformations with respect to e_0 of $L(V(e_0))$ is denoted by $R(V(e_0))$.

Lemma 24. *If $A \in R(V(e_0))$, then $A^{-1} \in R(V(e_0))$.*

Proof. We should show that A^{-1} is regular. In other words it should be proven that $A^{-1}e_0$ is a regular vector of $V(e_0)$. Now $A^{-1}e_0 = r(e_0) \vdash e_0 = \langle r(e_0), e_0\rangle e_0 = \left(\left\langle \frac{(e_0 - Ae_0)}{\langle Ae_0, e_0\rangle} + e_0, e_0 \right\rangle\right) e_0 = \frac{e_0}{\langle Ae_0, e_0\rangle}$. Hence, we have

$$\langle A^{-1}e_0, e_0\rangle = \frac{1}{\langle Ae_0, e_0\rangle},$$

thus $\langle A^{-1}e_0, e_0\rangle \neq 0$. The Lemma is proved. □

We also remember that in the work [3] on the space $L(V(e_0))$ a dialgebra structure was introduced using the following two products: Let us take A, B, C and D in $L(V(e_0))$ then are defined $A \vdash B$ and $C \dashv D$ in each x of $V(e_0)$ making $(A \vdash B)(x) = Ae_0 \vdash Bx$ and $(C \dashv D)(x) = Cx \dashv De_0$. These products convert to the space $L(V(e_0))$ in a dialgebra. Let us see it in detail, first of all note that $A \vdash B$ and $C \dashv D$ are linear operators for all A, B, C and D in $L(V(e_0))$. It follows of the definition of $\vdash$ that for A, B and C in $L(V(e_0))$

$$\begin{aligned}(A \dashv (B \dashv C))(x) &= Ax \dashv (B \dashv C)(e_0)\\ &= Ax \dashv Be_0 \dashv Ce_0,\end{aligned}$$

on the other hand

$$\begin{aligned}(A \dashv (B \vdash C))(x) &= Ax \dashv (B \vdash C)(e_0)\\ &= Ax \dashv Be_0 \vdash Ce_0,\end{aligned}$$

since $V(e_0)$ is a dialgebra then $Ax \dashv (Be_0 \dashv Ce_0) = Ax \dashv (Be_0 \vdash Ce_0)$. Therefore we have $A \dashv (B \dashv C) = (A \dashv (B \vdash C))$.

In a similar way one can show that $(A \vdash B) \dashv C = A \vdash (B \dashv C)$. In fact

$$\begin{aligned}((A \vdash B) \dashv C)(x) &= (A \vdash B)(x) \dashv Ce_0\\ &= (Ae_0 \vdash Bx) \dashv Ce_0,\end{aligned}$$

and we check

$$\begin{aligned}(A \vdash (B \dashv C))(x) &= Ae_0 \vdash (B \dashv C)(x)\\ &= Ae_0 \vdash (Bx \dashv Ce_0),\end{aligned}$$

using now the fact that $V(e_0)$ is a dialgebra we have $((A \vdash B) \dashv C) = (A \vdash (B \dashv C))$. The reader easily examines that also $(A \dashv B) \vdash C = (A \vdash B) \vdash C$.

The following Lemma will be useful.

Lemma 25. *Let A be a element of $L(V(e_0))$, regular relative to e_0 and let $R \in L(V(e_0))$ such that $R \dashv A = I = A \vdash R$. Then $R = A^{-1}$, that is, $Rx = r(e_0) \vdash x = \langle r(e_0), e_0 \rangle x = x \dashv r(e_0)$ for all $x \in V(e_0)$.*

Proof. Since $R \dashv A = I$, for any $x \in V(e_0)$ we have $x = (R \dashv A)x = Rx \dashv Ae_0$. Then if $r(e_0)$ is the inverse of Ae_0 in the dialgebra $V(e_0)$ we obtain

$$\begin{aligned} x &\dashv r(e_0) = (Rx \dashv Ae_0) \dashv r(e_0) \\ &= Rx \dashv (Ae_0 \dashv r(e_0)) \\ &= Rx \dashv (Ae_0 \vdash r(e_0)) \\ &= Rx \dashv ((e_0 - Ae_0) + (Ae_0 \vdash e_0)) \\ &= (Rx \dashv (e_0 - Ae_0)) + (Rx \dashv (Ae_0 \vdash e_0)) \\ &= (Rx \dashv (e_0 - Ae_0)) + (Rx \dashv (Ae_0 \dashv e_0)) \\ &= (Rx \dashv (e_0 - Ae_0)) + (Rx \dashv Ae_0) \\ &= (Rx \dashv e_0) - (Rx \dashv Ae_0) + (Rx \dashv Ae_0) \\ &= Rx, \end{aligned}$$

because $r(e_0) \vdash x = \langle r(e_0), e_0 \rangle x = x \dashv r(e_0)$, also it is easy to see that the equality $I = A \vdash R$ implies that $R = A^{-1}$. □

5 Principal linear representation of digroups

We have

Theorem 26. $(R(V(e_0)), I)$ *is a digroup.*

Proof. Note that if A, $B \in R(V(e_0))$ then $A \vdash B$, $A \dashv B \in R(V(e_0))$. Now, suppose that $A \in R(V(e_0))$ then from the Lemma 26 we have that $A^{-1} \in R(V(e_0))$ and according the Lemma 27 it follows that A^{-1} is unique. Since $L(V(e_0))$ is a dialgebra, we only need to verify that $I \in R(V(e_0))$, and moreover that $I \vdash A = A = A \dashv I$ for all $A \in R(V(e_0))$, but these facts are evident. □

It is the moment for the following definition

Definition 27. *Let* (D, e) *be a digroup and* V *a finite dimensional vector space and* $L(V(e_0))$ *the principal dialgebra generated by* V *and* e_0. *A principal linear representation of* (D, e) *by* V *is a transformation* $T : D \to R(V(e_0))$ *such that*

$$\begin{gathered} Te = I, \\ T(d_1 \vdash d_2) = Td_1 \vdash Td_2, \\ T(d_1 \dashv d_2) = Td_1 \dashv Td_2 \end{gathered} \tag{6}$$

It follows of $(6-2)$ and $(6-3)$ that $Tg \vdash Tg^{-1} = T\left(g \vdash g^{-1}\right) = Te = I$ moreover $Tg \dashv Tg^{-1} = T\left(g \dashv g^{-1}\right) = Te = I$, Hence $Tg^{-1} = (Tg)^{-1}$. The properties $(6-2)$ and $(6-3)$ imply that T is a homomorphism.

It clear that if T is a principal linear representation of (D, e) by V and (H, e) is a subdigroup of (D, e) then $T_{|H}$ is a principal linear representation of (H, e) by V. The mapping $Eg = I$ for any $g \in D$ is a principal linear representation and it is called the trivial principal linear representation.

Example 28. *$\varphi-$digroups and a principal linear representation for these. We recall the following facts (see Example 7): let V be a finite dimensional vector space and let φ be a nonzero element of V^*, then the set $V(\varphi) = \{x \in V \mid \varphi(x) \neq 0\}$ can be equipped with the following two products: the first $x \vdash y = \varphi(x) y$ and the second $z \dashv w = \varphi(w) z$ for any x, y, z and w belong to V. Let $x_0 \in V(\varphi)$ and $e = \frac{x_0}{\varphi(x_0)}$ (note that $e \in V(\varphi)$) then as it was shown $(V(\varphi), e)$ is a digroup relative to these two products which was called the $\varphi-$digroup generated by φ. We recall also that for $x \in V(\varphi)$ its inverse is the element $x^{-1} = \frac{e}{\varphi(x)}$. Next, we can then to construct a basis of V which contain the vector e. In this basis we have $\langle e, e\rangle = 1$. Thus, it is possible to introduce the space $L(V(e))$ itself. Let us define T_x for any $x \in V(\varphi)$ of the following form: $T_x z = x \vdash z$, evidently T_x is a linear transformation and is regular. To see this later observe that $T_x e = x \vdash e = \varphi(x) e$, thus $\langle T_x e, e\rangle = \varphi(x) \neq 0$ since $x \in V(\varphi)$. These transformations can be used to construct a principal linear representation of $V(\varphi)$ by V. In fact, $T_e = I$, that is, T_e is the identity operator on $V(e)$. On the other hand for any x, $y \in V(\varphi)$ and all $z \in V(e)$ we have $T_{(x \vdash y)} z = (x \vdash y) \vdash z = \varphi(x) \varphi(y) z$, moreover $(T_x \vdash T_y) z = T_x e \vdash T_y z = (\varphi(x) e) \vdash (\varphi(y) z) = \varphi(x) \varphi(e) \varphi(y) z = \varphi(x) \varphi(y) z$. Hence, $T_{(x \vdash y)} = T_x \vdash T_y$. The equality $T_{(x \dashv y)} = T_x \dashv T_y$ is proved in similar way.*

Now, we will indicate as to construct some principal linear representations for concrete finite digroups.

Formally, a digroup (D, e) is called finite if it have a finite number of elements. If (D, e) have n elements, we say that (D, e) is of order n. As already was indicated if $D = \{x, y\}$ is an arbitrary set of two elements we can introduce a 2×2 $(\vdash)-$multiplication table and a 2×2 $(\dashv)-$multiplication table in D of the following form:

$\vdash$	x	y
x	x	y
y	x	y

$\dashv$	x	y
x	x	x
y	y	y

,

such that (D, x) is a digroup of order 2. This digroup will be denoted by $D(x, y)$.

A principal linear representation of $D(x, y)$ is constructed as follows. Let V be a vector space of dimension 2 in which we have fixed a basis $\{v_1, v_2\}$. Let e_0 be the following vector of V : $e_0 = \frac{1}{\sqrt{2}}(v_1 + v_2)$, is clear that $\|e_0\|^2 = 1$. This linear principal representation must be defined by two linear transformations of $L(V(e_0))$. We put $T_x = I$, where I is the identity transformation of $L(V(e_0))$ and define T_y as the linear transformation of $L(V(e_0))$ which sends v_1 to v_2 and sends v_2 to v_1. We wish to prove that $T : x \to T_x$, $y \to T_y$ is a principal linear representation of $D(x, y)$ by V. In fact, first of all we observe that $T_y e_0 = \frac{1}{\sqrt{2}}(T_y v_1 + T_y v_2) = \frac{1}{\sqrt{2}}(v_2 + v_1) = e_0$. It follows that $\langle T_y e_0, e_0\rangle =$

$1 \neq 0$. Hence, T_y is a regular linear transformations with respect to e_0. It is obvious that $T_{x\vdash y} = T_y$, on the other hand for all $z \in V(e_0)$ we have $(T_x \vdash T_y)\, z = (I \vdash T_y)\, z = e_0 \vdash T_y z = T_y z$. It shows that $T_{x\vdash y} = T_x \vdash T_y$. We must also verify that $T_{y\vdash x} = T_y \vdash T_x$. Note that $T_{y\vdash x} = T_x = I$ and moreover for any $z \in V(e_0)$ we have $(T_y \vdash T_x)\, z = T_y e_0 \vdash T_x z = e_0 \vdash z = z = Iz$. Consequently it implies that $T_y \vdash T_x = I$, thus $T_{y\vdash x} = T_y \vdash T_x$. Finally, it is easy to see that $T_{x\dashv y} = T_x = I$, $T_{y\dashv x} = T_y$, $T_x \dashv T_y = I \dashv T_y = I$ and $T_y \dashv T_x = T_y \dashv I = T_y$. Hence $T_{x\dashv y} = T_x \dashv T_y$ and $T_{y\dashv x} = T_y \dashv T_x$. Now we shall see what happen in this construction for products $x \vdash x$, $x \dashv x$, $y \vdash y$ and $y \dashv y$. Note that $x \vdash x = x = x \dashv x$ and $y \vdash y = y = y \dashv y$ then we have $T_{x\vdash x} = T_{x\dashv x} = T_x = I$, on the other hand $T_x \vdash T_x = T_x \dashv T_x = I$. By a simple calculation we also have $T_{y\vdash y} = T_{y\dashv y} = T_y$ and since $T_y e_0 = e_0$, $T_y \vdash T_y = T_y = T_y \dashv T_y$. Thus, we have constructed a principal linear representation of $D(x, y)$ by this V.

This construction can be extended to certain class of finite digroups (see Example 5 when X is a finite set). A "plucked" digroup is a digroup (D, e) in which the products $\vdash$ and $\dashv$ are defined in the following form: $x \vdash y = y$ for all x, $y \in D$ and $z \dashv w = z$ for any z, $w \in D$. If a "plucked" digroup is of finite order and its order is n we denote it by $D(x_1, \cdots, x_n)$ where $D = \{x_1, \cdots, x_n\}$ is the set of its elements and x_1 is the bar-unit of the digroup. Now we discuss here a principal linear representation of $D(x_1, \cdots, x_n)$; as before we take a vector space V of equal dimension that the order of the digroup, that is, $\dim V = n$. We can also choose a basis $\{v_{x_i}\}_{x_i \in D}$ indexed by the elements x_i of $D(x_1, \cdots, x_n)$. Put $e_0 = \frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n} v_{x_i}\right)$ then as before $\|e_0\|^2 = 1$. We define $T_{x_1} = I$ and for each x_i, $i \neq 1$ we define T_{x_i} as the linear transformation of $L(V(e_0))$ which sends v_{x_i} to v_{x_1} and v_{x_1} to v_{x_i} moreover it sends v_{x_k} to v_{x_k} for any k different of 1 and i. We use the following simple Lemma

Lemma 29. *For any $x_i \in D$, the map T_{x_i} is regular, in other words $T_{x_i} \in R(V(e_0))$.*

Proof. In fact, we have seen above that $T_{x_1} = I$ is regular. For any $x_i \in D$ with $i \neq 1$ we have

$$\begin{aligned} T_{x_i} e_0 &= \frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n} T v_{x_i}\right) \\ &= \frac{1}{\sqrt{n}}\left(\left(\sum_{j\neq 1,\ j\neq i} T v_{x_j}\right) + T v_{x_1} + T v_{x_i}\right) \\ &= \frac{1}{\sqrt{n}}\left(\left(\sum_{j\neq 1,\ j\neq i} v_{x_j}\right) + v_{x_j} + v_{x_1}\right) \\ &= e_0, \end{aligned}$$

it follows that $\langle T_{x_i}e_0, e_0\rangle = 1 \neq 0$. Hence T_{x_i} is regular also for $i \neq 1$. $\square$

Let us now apply this Lemma to present principal linear representations of the digroup $D(x_1, \cdots, x_n)$. They are constructed in similar way that in the case $D(x, y)$.

Theorem 30. *Let $D(x_1, \cdots, x_n)$ be the "plucked" digroup defined by the set $D = \{x_1, \cdots, x_n\}$. A principal representation of $D(x_1, \cdots, x_n)$ by V is given for the map $T : x_i \to T_{x_i}$. Here V is a vector space of dimension n and T_{x_i} are the linear transformations constructed as before in the space $L(V(e_0))$.*

Proof. ¿From Lemma 31 the linear transformations T_{x_k} are regulars for all k. The prove of $T_{x_1 \vdash x_i} = T_{x_1} \vdash T_{x_i}$, $T_{x_i \vdash x_1} = T_{x_i} \vdash T_{x_1}$, $T_{x_1 \dashv x_i} = T_{x_1} \dashv T_{x_i}$ and $T_{x_i \dashv x_1} = T_{x_i} \dashv T_{x_1}$ for $i \neq 1$ is similar to the case $n = 2$. Let $i \neq 1$ and $j \neq 1$ then clearly $T_{x_i \vdash x_j} = T_{x_j}$. Now, we observe that for all $z \in V(e_0)$ from the proof of the Lemma 31 it is holds that $\left(T_{x_i} \vdash T_{x_j}\right) z = T_{x_i}e_0 \vdash T_{x_j}z = e_0 \vdash T_{x_j}z = T_{x_j}z$, that is $T_{x_i} \vdash T_{x_j} = T_{x_j}$. It follows that $T_{x_i \vdash x_j} = T_{x_i} \vdash T_{x_j}$. In similar way is proved that $T_{x_i \dashv x_j} = T_{x_i} \dashv T_{x_j}$ for all i and j such that $i \neq 1$ and $j \neq 1$. Finally it is evident that $T_{x_1 \vdash x_1} = T_{x_1} \vdash T_{x_1} = I$ and $T_{x_1 \dashv x_1} = T_{x_1} \dashv T_{x_1} = I$. $\square$

6 Linear representation of digroups by abelian dialgebras

Let $(V, \vdash, \dashv)$ be a dialgebra and let e_0 be a bar-unit of it. The subset of $L(V)$ be composed of all the linear transformations which are regular with respect to e_0, is denoted by $R(V, e_0)$. Explicitly, $A \in R(V, e_0)$ means that $A \in L(V)$ and Ae_0 is regular in V. It is well known that $L(V)$ is a dialgebra relative to the same products that were introduced above for the space $L(V(e_0))$. There V was a finite dimensional vector space but this condition is not essential (see [3]).We say that $(V, \vdash, \dashv)$ is abelian if $y \vdash x = x \dashv y$ for all x, $y \in V$, clearly it does not means that $\vdash = \dashv$.

We start this part of the paper extending some results of the section 5. We first observe that if A, $B \in R(V, e_0)$ where V is an abelian dialgebra, then $A \vdash B$, $A \dashv B \in R(V, e_0)$. In fact, assume that A, $B \in R(V, e_0)$, since Ae_0 and Be_0 are regulars in V, then

$$Ae_0 \vdash r_1(e_0) = (e_0 - Ae_0) + (Ae_0 \vdash e_0),$$

and

$$Be_0 \vdash r_2(e_0) = (e_0 - Be_0) + (Be_0 \vdash e_0),$$

where $r_1(e_0)$, $r_2(e_0) \in V$.

Let us denote $h(e_0) = (e_0 - (Ae_0 \vdash Be_0)) + ((Ae_0 \vdash Be_0) \vdash e)$. It is easy to see that

$$\begin{aligned} Ae_0 &\vdash (Be_0 \vdash r_2(e_0)) = (Ae_0 \vdash Be_0) \vdash r_2(e_0) \\ &= (Ae_0 \vdash e_0) - e_0 + h(e_0), \end{aligned}$$

but $(Ae_0 \vdash e_0) - e_0 = Ae_0 \vdash k(e_0)$, where $k(e_0) = e_0 - (r_1(e_0) \vdash e_0)$. Thus

$$(Ae_0 \vdash Be_0) \vdash r_2(e_0) = (Ae_0 \vdash Be_0) \vdash (r_2(e_0) \vdash k(e_0)) + h(e_0),$$

hence $(Ae_0 \vdash Be_0) \vdash (r_2(e_0) - (r_2(e_0) \vdash k(e_0))) = h(e_0)$. Now, since V is an abelian dialgebra

$$z(e_0) \dashv (Ae_0 \vdash Be_0) = (e_0 - (Ae_0 \vdash Be_0)) + (e_0 \dashv (Ae_0 \vdash Be_0)),$$

where $z(e_0) = (r_2(e_0) - (r_2(e_0) \vdash k(e_0)))$. It follows that $(A \vdash B)e_0 = Ae_0 \vdash Be_0$ is regular in V. Finally, observe that being V an abelian dialgebra $(A \dashv B)e_0 = (Ae_0 \dashv Be_0) = (Be_0 \vdash Ae_0)$ and as it was already seen $(Be_0 \vdash Ae_0)$ is regular in V, hence $(A \dashv B)e_0$ is regular in V. Then also $A \dashv B \in R(V, e_0)$ if A, $B \in R(V, e_0)$.

Theorem 31. *Let $(V, \vdash, \dashv)$ be an abelian dialgebra. Then $(R(V, e_0), I_V)$ is a digroup.*

Proof. If $A \in R(V, e_0)$ then by definition Ae_0 is vector regular in $(V, \vdash, \dashv)$. Hence, there is a vector $r(e_0) \in V$ which is $(\vdash)$-inverse and also $(\dashv)$-inverse of Ae_0. It allows to construct a element of $L(V)$ of the following form $A^{-1}x = r(e_0) \vdash x$ for all $x \in V$. Since $(V, \vdash, \dashv)$ is abelian, in his turn one have that $A^{-1}x = x \dashv r(e_0)$. Now, we know that for any $x \in V$

$$(Ae_0 \vdash r(e_0)) \vdash x = x = x \dashv (r(e_0) \dashv Ae_0),$$

it follows that $A \vdash A^{-1} = I_V = A^{-1} \dashv A$. Let us to show that A^{-1} is regular relative to e_0. In fact, we must find a vector y such that $A^{-1}e_0 \vdash y = \left(e_0 - A^{-1}e_0\right) + \left(A^{-1}e_0 \vdash e_0\right)$ and $y \dashv A^{-1}e_0 = \left(e_0 - A^{-1}e_0\right) + \left(e_0 \dashv A^{-1}e_0\right)$. We assume that such vector exists. Then, since $A^{-1}e_0 = r(e_0) \vdash e_0$ we have that $r(e_0) \vdash y = e_0$, thus $Ae_0 \vdash (r(e_0) \vdash y) = Ae_0 \vdash e_0$. Hence, $y = Ae_0 \vdash e_0$ because $\vdash$ is associative. Next, we prove that $y = Ae_0 \vdash e_0$ is in fact $(\dashv)$-inverse of $A^{-1}e_0$. First of all, note that $y = e_0 \dashv Ae_0$

$$\begin{aligned} (e_0 \dashv Ae_0) &\dashv A^{-1}e_0 = e_0 \dashv \left(Ae_0 \dashv A^{-1}e_0\right) \\ &= e_0 \dashv \left(Ae_0 \vdash A^{-1}e_0\right) \\ &= e_0 \dashv ((e_0 - Ae_0) + (Ae_0 \vdash e_0)) \\ &= e_0 - (e_0 \dashv Ae_0) + (e_0 \dashv (Ae_0 \vdash e_0)) \\ &= e_0, \end{aligned}$$

and

$$\begin{aligned}\left(e_0 - A^{-1}e_0\right) + \left(e_0 \dashv A^{-1}e_0\right) &= \left(e_0 - \left(e_0 \dashv r\left(e_0\right)\right)\right) + \left(e_0 \dashv \left(e_0 \dashv r\left(e_0\right)\right)\right)\\ &= \left(e_0 - \left(e_0 \dashv r\left(e_0\right)\right)\right) + \left(e_0 \dashv r\left(e_0\right)\right)\\ &= e_0,\end{aligned}$$

thus, we have proved that $y \dashv A^{-1}e_0 = \left(e_0 - A^{-1}e_0\right) + \left(e_0 \dashv A^{-1}e_0\right)$. On the other it is possible to show that if $R \in R\left(V, e_0\right)$ is such that $A \vdash R = I_V = R \dashv A$ it imply $R = A^{-1}$. Finally, observe that $I_V \in R\left(V, e_0\right)$ and $I_V \vdash A = A = A \dashv I_V$ for all $A \in R\left(V, e_0\right)$. □

Definition 32. *Let (D, e) be a digroup and let $(V, \vdash, \dashv)$ be an abelian dialgebra. A linear representation of (D, e) by V (note that V is a vector space) is a transformation $T : D \to R\left(V, e_0\right)$ such that*

$$\begin{gathered}Te = I,\\ T\left(d_1 \vdash d_2\right) = Td_1 \vdash Td_2,\\ T\left(d_1 \dashv d_2\right) = Td_1 \dashv Td_2\end{gathered} \tag{7}$$

From (7) it shows that T is a homomorphism.

Let D_1 and D_2 be two dialgebras with products denoted the equal form $\vdash$, $\dashv$. Their direct sum $D_1 \oplus D_2$ is defined as the Cartesian product $D_1 \times D_2$ with the coordinate-wise operations. It is easy to show that the direct sum $D_1 \oplus D_2$, so is a dialgebra with products defined of the following form: if (d_1, d_2), $(f_1, f_2) \in D_1 \oplus D_2$, then one defines $(d_1, d_2) \star (f_1, f_2) = (d_1 \star f_1, d_2 \star f_2)$ for $\star = \vdash$ and $\dashv$. Observe that being e_1 and e_2 bar-unit of D_1 and D_2 respectively, then (e_1, e_2) is a bar-unity of $D_1 \oplus D_2$. Also note that if D_1 and D_2 are abelian dialgebras then $D_1 \oplus D_2$ is an abelian dialgebra. In fact, let (d_1, d_2), (f_1, f_2) be vectors of $D_1 \oplus D_2$ then $(d_1, d_2) \vdash (f_1, f_2) = (d_1 \vdash f_1, d_2 \vdash f_2) = (f_1 \dashv d_1, f_2 \dashv d_2) = (f_1, f_2) \dashv (d_1, d_2)$.

Let V and W be two finite vectors space. Let $\{v_1, \cdots, v_n\}$ and $\{w_1, \cdots, w_m\}$ be basis of V and W respectively. Consider the vectors $\widehat{e}$, $\widetilde{e}$ where $\widehat{e} \in V$ and $\widetilde{e} \in W$, for which in these basis $\|\widehat{e}\|^2 = \|\widetilde{e}\|^2 = 1$. Then, it is clear that we can define the dialgebras $V(\widehat{e})$ and $W(\widetilde{e})$, and so $V(\widehat{e}) \oplus W(\widetilde{e})$ as above. Moreover, the vector $(\widehat{e}, \widetilde{e})$ is a bar-unit of $V(\widehat{e}) \oplus W(\widetilde{e})$. Since that $V(\widehat{e})$ and $W(\widetilde{e})$ are abelian dialgebras then $V(\widehat{e}) \oplus W(\widetilde{e})$ is also an abelian dialgebra.

Let $A \in L(V(\widehat{e}))$ and $B \in L(W(\widetilde{e}))$. Let $A \oplus B$ be the element of $L(V(\widehat{e}) \oplus W(\widetilde{e}))$ defined of the following form: $(A \oplus B)(x, y) = (Ax, By)$ for any $(x, y) \in V(\widehat{e}) \oplus W(\widetilde{e})$. Observe that $I_{V(\widehat{e})} \oplus I_{W(\widetilde{e})} = I_{V(\widehat{e}) \oplus W(\widetilde{e})}$. We say that $A \oplus B$ is feasible if A and B are regulars in $\widehat{e}$ and $\widetilde{e}$ respectively.

Proposition 33. *Let $A \oplus B$ be feasible. Then $A \oplus B \in R(V(\widehat{e}) \oplus W(\widetilde{e}))$.*

Proof. We shall prove that $A \oplus B$ is regular relative to $(\widehat{e}, \widetilde{e})$, that is, that $(A \oplus B)(\widehat{e}, \widetilde{e})$ is regular in $V(\widehat{e}) \oplus W(\widetilde{e})$. Since that A and B are regulars in $\widehat{e}$ and $\widetilde{e}$ respectively, there is $(r(\widehat{e}), r(\widetilde{e})) \in V(\widehat{e}) \oplus W(\widetilde{e})$ such that

$$A\widehat{e} \vdash r(\widehat{e}) = (\widehat{e} - A\widehat{e}) + (A\widehat{e} \vdash \widehat{e}), \quad B\widetilde{e} \vdash r(\widetilde{e}) = (\widetilde{e} - B\widetilde{e}) + (B\widetilde{e} \vdash \widetilde{e}), \tag{8}$$

and

$$r(\widehat{e}) \dashv A\widehat{e} = (\widehat{e} - A\widehat{e}) + (\widehat{e} \dashv A\widehat{e}), \quad r(\widetilde{e}) \dashv B\widetilde{e} = (\widetilde{e} - B\widetilde{e}) + (\widetilde{e} \dashv B\widetilde{e}). \tag{9}$$

Hence, we have from (8)

$$\begin{aligned}(A \oplus B)(\widehat{e}, \widetilde{e}) \ \vdash \ & (r(\widehat{e}), r(\widetilde{e})) = (A\widehat{e}, B\widetilde{e}) \vdash (r(\widehat{e}), r(\widetilde{e})) \\ = \ & (A\widehat{e} \vdash r(\widehat{e}), B\widetilde{e} \vdash r(\widetilde{e})) \\ = \ & ((\widehat{e} - A\widehat{e}) + (A\widehat{e} \vdash \widehat{e}), (\widetilde{e} - B\widetilde{e}) + (B\widetilde{e} \vdash \widetilde{e})) \\ = \ & ((\widehat{e}, \widetilde{e}) - (A\widehat{e}, B\widetilde{e})) + ((A\widehat{e}, B\widetilde{e}) \vdash (\widehat{e}, \widetilde{e})) \\ = \ & ((\widehat{e}, \widetilde{e}) - (A \oplus B)(\widehat{e}, \widetilde{e})) + ((A \oplus B)(\widehat{e}, \widetilde{e}) \vdash (\widehat{e}, \widetilde{e})),\end{aligned}$$

it follows that $(A \oplus B)(\widehat{e}, \widetilde{e})$ is $(\vdash)$-regular in $V(\widehat{e}) \oplus W(\widetilde{e})$. In similar way from (9), one can see that

$$(r(\widehat{e}), r(\widetilde{e})) \dashv (A \oplus B)(\widehat{e}, \widetilde{e}) = ((\widehat{e}, \widetilde{e}) - (A \oplus B)(\widehat{e}, \widetilde{e})) + ((\widehat{e}, \widetilde{e}) \dashv (A \oplus B)(\widehat{e}, \widetilde{e})).$$

Thus, also $(A \oplus B)(\widehat{e}, \widetilde{e})$ is $(\dashv)$-regular and moreover note that $(r(\widehat{e}), r(\widetilde{e}))$ is both $(\vdash)$-inverse and $(\dashv)$-inverse for the vector $(A \oplus B)(\widehat{e}, \widetilde{e})$. Hence $(A \oplus B)(\widehat{e}, \widetilde{e})$ is regular in $V(\widehat{e}) \oplus W(\widetilde{e})$. It is shows that $(A \oplus B)$ is regular with respect to $(\widehat{e}, \widetilde{e})$. Thus $A \oplus B \in R(V(\widehat{e}) \oplus W(\widetilde{e}))$. The Lemma is proved. ☐

We are now in condition of to prove the following result

Theorem 34. *Let (D, e) be a digroup. Let $\widehat{T}$ and $\widetilde{T}$ be two principal linear representations of (D, e) by two finite dimensional vector spaces V and W respectively, where $\widehat{T} : D \to R(V(\widehat{e}))$ and $\widetilde{T} \to R(W(\widetilde{e}))$. Let us define the map $\widehat{T} \oplus \widetilde{T}$ as follows $\left(\widehat{T} \oplus \widetilde{T}\right) d = \widehat{T}d \oplus \widetilde{T}d$ for all $d \in D$. Under the condition: $\widehat{T}d \oplus \widetilde{T}d$ is feasible for all $d \in G$, then $\widehat{T} \oplus \widetilde{T}$ is a linear representation of (D, e) by $V(\widehat{e}) \oplus W(\widetilde{e})$. It is called the direct sum of $\widehat{T}$ and $\widetilde{T}$.*

Proof. It is clear that $\left(\widehat{T} \oplus \widetilde{T}\right) e = \widehat{T}e \oplus \widetilde{T}e = I_{V(\widehat{e})} \oplus I_{W(\widetilde{e})} = I_{V(\widehat{e}) \oplus W(\widetilde{e})}$. We are going to show that $\left(\widehat{T} \oplus \widetilde{T}\right)(d \vdash g) = \left(\widehat{T} \oplus \widetilde{T}\right) d \vdash \left(\widehat{T} \oplus \widetilde{T}\right) g$ for all d, $g \in D$. In fact, since $\widehat{T}$ and $\widetilde{T}$ are principal linear representations of (D, e)

by V and W respectively we have $\left(\widehat{T}\oplus\widetilde{T}\right)(d\vdash g)=\widehat{T}(d\vdash g)\oplus\widetilde{T}(d\vdash g)=\left(\widehat{T}d\vdash\widehat{T}g\right)\oplus\left(\widetilde{T}d\vdash\widetilde{T}g\right)$. Hence for all $(x,y)\in V(\widehat{e})\oplus W(\widetilde{e})$, it follows

$$\begin{aligned}\left(\left(\widehat{T}\oplus\widetilde{T}\right)(d\vdash g)\right)(x,y) &= \left(\left(\widehat{T}d\vdash\widehat{T}g\right)\oplus\left(\widetilde{T}d\vdash\widetilde{T}g\right)\right)(x,y)\\ &= \left(\left(\widehat{T}d\vdash\widehat{T}g\right)x,\left(\widetilde{T}d\vdash\widetilde{T}g\right)y\right)\\ &= \left(\left((\widehat{T}d)\widehat{e}\vdash(\widehat{T}g)x\right),\left((\widetilde{T}d)\widetilde{e}\vdash(\widetilde{T}g)y\right)\right)\\ &= \left((\widehat{T}d)\widehat{e},(\widetilde{T}d)\widetilde{e}\right)\vdash\left((\widehat{T}g)x,(\widetilde{T}g)y\right)\\ &= \left((\widehat{T}d)\oplus(\widetilde{T}d)\right)(\widehat{e},\widetilde{e})\vdash\left((\widehat{T}g)\oplus(\widetilde{T}g)\right)(x,y)\\ &= \left(\left(\widehat{T}\oplus\widetilde{T}\right)d\right)(\widehat{e},\widetilde{e})\vdash\left(\left(\widehat{T}\oplus\widetilde{T}\right)g\right)(x,y)\\ &= \left(\left(\left(\widehat{T}\oplus\widetilde{T}\right)d\right)\vdash\left(\left(\widehat{T}\oplus\widetilde{T}\right)g\right)\right)(x,y),\end{aligned}$$

thus $\left(\left(\widehat{T}\oplus\widetilde{T}\right)(d\vdash g)\right)=\left(\left(\widehat{T}\oplus\widetilde{T}\right)d\right)\vdash\left(\left(\widehat{T}\oplus\widetilde{T}\right)g\right)$. Here we have used the fact that $\left((\widehat{T}d)\widehat{e},(\widetilde{T}d)\widetilde{e}\right),\left((\widehat{T}g)x,(\widetilde{T}g)y\right)\in V(\widehat{e})\oplus W(\widetilde{e})$. The proof that $\left(\left(\widehat{T}\oplus\widetilde{T}\right)(d\dashv g)\right)=\left(\left(\widehat{T}\oplus\widetilde{T}\right)d\right)\dashv\left(\left(\widehat{T}\oplus\widetilde{T}\right)g\right)$ is similar □

We recall that two digroups (D_1,e_1) and (D_2,e_2) are said to be homomorphic if there exists a mapping $\gamma:D_1\to D_2$ such that $\gamma(x\vdash y)=\gamma(x)\vdash\gamma(y)$ and $\gamma(z\dashv w)=\gamma(z)\dashv\gamma(w)$ for all z, y, z, $w\in D_1$. We say that D_1 and D_2 are isomorphic if it is proved that γ is one to one.

Theorem 35. *Let γ be a homomorphic of (D_1,e_1) into (D_2,e_2) such that $\gamma(e_1)=e_2$. Let T be linear representation of (D_2,e_2) by V, where $(V,\vdash,\dashv)$ is an abelian dialgebra. Then, $T\circ\gamma$ is a linear representation of (D_1,e_1) by V*

Proof. Because $(\gamma(D_1),e_2)$ is subdigroup of (D_2,e_2) the Theorem is evident. □

We remark in closing of this section that the next step is to decide if representations of digroups are essentially just representations of their groups of inverses or if there is more to them.

7 The digroup dialgebra

Now we come to the construction of the "digroup dialgebra" of an arbitrary finite digroup (D,e).

Let (D,e) be a finite digroup. Consider the set of all formal finite sums $a=\sum_{d\in D}a(d)\,d$ where $a(d)\in\mathbb{C}$ for all $d\in D$, two such expressions being regarded as equal if and only if they have the same coefficients. Clearly, this set is a complex vector space, where $\alpha a = \sum_{d\in D} \alpha a\,(d)\, d$ for any $\alpha \in \mathbb{C}$ and $a + b = \sum_{d\in D} (a\,(d) + b\,(d))d$ if $b = \sum_{d\in D} b\,(d)\, d$, $b\,(d) \in \mathbb{C}$ for all $d \in D$. The zero of this vector space is the formal sum having all the coefficients equal to zero. This vector space will be denoted by $\mathcal{L}\,(D)$. We would like to give $\mathcal{L}\,(D)$ two bilinear products. Thus, we define for $a,\, b \in \mathcal{L}\,(D)$

$$a \vdash b = \left(\sum_{d\in D} a\,(d)\, d\right) \vdash \left(\sum_{g\in D} b\,(g)\, g\right) = \sum_{d,g} a\,(d)\, b\,(g)\,(d \vdash g)\,, \tag{10}$$

and

$$a \dashv b = \left(\sum_{d\in D} a\,(d)\, d\right) \dashv \left(\sum_{g\in D} b\,(g)\, g\right) = \sum_{d,g} a\,(d)\, b\,(g)\,(d \dashv g)\,, \tag{11}$$

Proposition 36. $(\mathcal{L}\,(D)\,, \vdash, \dashv)$ *is a dialgebra.*

Proof. Since the bilinearity of the products (10) and (11) is evident, the Proposition follows from the properties of digroup products. □

Let $f_d = \sum_g \delta_{dg} g$, where δ_{dg} is the function that is equal to 1 if $g = d$ and 0 for $g \neq d$. Note that formally $f_d = d$ for any $d \in D$. Thus, we can consider that D is contained in $\mathcal{L}\,(D)$ by to identify each $d \in D$ with the sum f_d corresponding. Also observe that f_e is a bar-unity of $\mathcal{L}\,(D)$, moreover the dimension of $\mathcal{L}\,(D)$ coincide with the order of (D, e). The dialgebra $\mathcal{L}\,(D)$ is called the "digroup dialgebra" associate to the digroup (D, e).

Remark 37. *Observe that* $(\mathcal{L}\,(D)\,, \vdash, \dashv)$ *is not an abelian dialgebra if the digroup* (D, e) *is not abelian. In fact, since for all* $d_1,\, d_2 \in D$, $f_{d_1} \vdash f_{d_2} = d_1 \vdash d_2 = f_{d_1 \vdash d_2}$ *and* $f_{d_2} \dashv f_{d_1} = d_2 \dashv d_1 = f_{d_2 \dashv d_1}$, *then* $(\mathcal{L}\,(D)\,, \vdash, \dashv)$ *is abelian if and only if* (D, e) *is an abelian digroup.*

Definition 38. *Let* $(\mathcal{D}, \vdash, \dashv)$ *be a complex dialgebra. A mapping* $x \to x^*$ *of* $\mathcal{D}$ *onto itself is called a reduced involution provided the following conditions are satisfied for all* $x,\, y \in D$ *and* $\alpha \in \mathbb{C}$*:*

$$\begin{array}{lc} (i) & (x + y)^* = x^* + y^*, \\ (ii) & (x \vdash y)^* = y^* \dashv x^*, \\ (iii) & (\alpha x)^* = \overline{\alpha} x^*, \end{array}$$

note that from (ii) *it follows the following equality*

$$(x \dashv y)^* = y^* \vdash x^*,$$

in this case, we say that $\mathcal{D}$ *is a reduced* $*$*-dialgebra.*

We have the following result

Theorem 39. *Let* $a = \sum_{d \in D} a(d)\, d$ *be an element of* $\mathcal{L}(D)$. *We define* $a^* = \sum_{d \in D} \overline{a(d)} d^{-1}$. *Then* $\mathcal{L}(D)$ *is a reduced* $*$*-dialgebra under the reduced involution:* $a \to a^*$. *In this case,*

$$(a^*)^* = f_e \dashv a = a \vdash f_e \tag{12}$$

Proof. It is easy to see that the equalities $(\alpha a)^* = \overline{\alpha} a^*$ and $(a+b)^* = a^* + b^*$ are always truth for all $\alpha \in \mathbb{C}$ and all $a, b \in \mathcal{L}(D)$. Now,

$$\begin{aligned} (a \vdash b)^* &= \left(\sum_{d,g} a(d)\, b(g)\, (d \vdash g) \right)^* \\ &= \left(\sum_{d,g} \overline{a(d)}.\overline{b(g)}\, (d \vdash g)^{-1} \right) \\ &= \left(\sum_{g,d} \overline{b(g)}.\overline{a(d)} \left(g^{-1} \vdash d^{-1} \right) \right) \\ &= \left(\sum_{g} \overline{b(g)} g^{-1} \right) \vdash \left(\sum_{d} \overline{a(d)} d^{-1} \right) \\ &= b^* \vdash a^*, \end{aligned}$$

Finally, we must to prove the relation (12):

$$\begin{aligned} (a^*)^* &= \left(\sum_{d \in D} \overline{a(d)} d^{-1} \right)^* \\ &= \left(\sum_{d \in D} a(d) \left(d^{-1} \right)^{-1} \right) \\ &= \left(\sum_{d \in D} a(d)\, (e \dashv d) \right) \\ &= f_e \dashv a, \end{aligned}$$

where we have used the fact that $\left(d^{-1}\right)^{-1} = (e \dashv d)$. Since, in a digroup (D, e) we have $(e \dashv d) = (d \vdash e) = \left(d^{-1}\right)^{-1}$ for any $d \in D$, it follows also that $(a^*)^* = a \vdash f_e$ as was to be proven. ☐

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