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ON QUASI-BEAR RINGS OF ORE EXTENSIONS

L'moufadal Ben Yakoub and **Mohamed Louzari**

Department of Mathematics,
University Abdelmalek Essaadi
B.P. 2121 Tetouan, Morocco
benyakoub@hotmail.com   louzari_mohamed@hotmail.co

Abstract

Let R be a ring and $S = R[x;\sigma,\delta]$ its Ore extension. We prove under some conditions that R is a quasi-Baer ring if and only if the Ore extension $R[x;\sigma,\delta]$ is a quasi-Baer ring. Examples are provided to illustrate and delimit our results.

1 Introduction

Throughout this paper, R denotes an associative ring with unity. For a subset X of R, $r_R(X) = \{a \in R | Xa = 0\}$ and $\ell_R(X) = \{a \in R | aX = 0\}$ will stand for the right and the left annihilator of X in R respectively. By [9], a right annihilator of X is always a right ideal, and if X is a right ideal then $r_R(X)$ is a two-sided ideal. An Ore extension of a ring R is denoted by $R[x;\sigma,\delta]$, where σ is an endomorphism of R and δ is a σ-derivation, i.e., $\delta\colon R \to R$ is an additive map such that $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$ for all $a, b \in R$. Recall that elements of $R[x;\sigma,\delta]$ are polynomials in x with coefficients written on the left. Multiplication in $R[x;\sigma,\delta]$ is given by the multiplication in R and the condition $xa = \sigma(a)x + \delta(a)$, for all $a \in R$. We say that a subset X of R is (σ,δ)-*stable* if $\sigma(X) \subseteq X$ and $\delta(X) \subseteq X$. A ring R is (*quasi*)-*Baer* if the right annihilator of every nonempty subset (every right ideal) of R is generated by an idempotent. From [1], an idempotent $e \in R$ is left (resp. right) *semicentral* in R if $exe = xe$ (resp. $exe = ex$), for all $x \in R$. Equivalently, $e^2 = e \in R$ is left (resp. right)

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semicentral if eR (resp. Re) is an ideal of R. Since the right annihilator of a right ideal is an ideal, we see that the right annihilator of a right ideal is generated by a left semicentral in a quasi-Baer ring. We use $\mathcal{S}_\ell(R)$ and $\mathcal{S}_r(R)$ for the sets of all left and right semicentral idempotents, respectively. Also note $\mathcal{S}_\ell(R) \cap \mathcal{S}_r(R) = \mathcal{B}(R)$, where $\mathcal{B}(R)$ is the set of all central idempotents of R. If R is a semiprime ring then $\mathcal{S}_\ell(R) = \mathcal{S}_r(R) = \mathcal{B}(R)$. Recall that R is a *reduced* ring if it has no nonzero nilpotent elements. A ring R is *abelian* if every idempotent of R is central. We can easily observe that every reduced ring is abelian.

According to [10], an endomorphism σ of a ring R is said to be *rigid* if $a\sigma(a) = 0$ implies $a = 0$ for all $a \in R$. We call a ring R *σ-rigid* if there exists a rigid endomorphism σ of R. Following Hashemi and Moussavi [4], a ring R is *σ-compatible* if for each $a, b \in R$, $a\sigma(b) = 0 \Leftrightarrow ab = 0$. Moreover, R is said to be *δ-compatible* if for each $a, b \in R$, $ab = 0 \Rightarrow a\delta(b) = 0$. If R is both σ-compatible and δ-compatible, we say that R is *(σ,δ)-compatible.* A ring R is σ-rigid if and only if R is (σ,δ)-compatible and reduced [4, Lemma 2.2]. Also, if R is σ-rigid then $R[x;\sigma,\delta]$ is reduced [10, Theorem 3.3]. From [8], a ring R is said to be a *σ-skew Armendariz* ring if for $p = \sum_{i=0}^{n} a_i x^i$ and $q = \sum_{j=0}^{m} b_j x^j$ in $R[x;\sigma]$, $pq = 0$ implies $a_i\sigma^i(b_j) = 0$ for all $0 \le i \le n$ and $0 \le j \le m$. From [5], a ring R is called an *(σ,δ)-skew Armendariz* ring if for $p = \sum_{i=0}^{n} a_i x^i$ and $q = \sum_{j=0}^{m} b_j x^j$ in $R[x;\sigma,\delta]$, $pq = 0$ implies $a_i x^i b_j x^j = 0$ for each i, j. Note that (σ,δ)-skew Armendariz rings are generalization of σ-skew Armendariz rings, σ-rigid rings and Armendariz rings, see [8], for more details. It was proved in [7, Corollary 12], that if R is a σ-rigid ring then $R[x;\sigma,\delta]$ is a quasi-Baer ring if and only if R is quasi-Baer. Also in [4, Corollary 2.8], it was shown that, if R is (σ,δ)-compatible, then $R[x;\sigma,\delta]$ is a quasi-Baer ring if and only if R is quasi-Baer.

The aim of this paper is to show that if R is an (σ,δ)-skew Armendariz ring with σ an automorphism such that Re is (σ,δ)-stable for all $e \in \mathcal{S}_\ell(R)$, then R is a quasi-Baer ring if and only if $R[x;\sigma,\delta]$ is a quasi-Baer ring. Many examples are provided to illustrate and delimit results and to show that they are not consequences of [4, Corollary 2.8]. Moreover, we obtain a partial generalization of [7, Corollary 12].

2 Preliminaries and Examples

For any $0 \le i \le j$ $(i, j \in \mathbb{N})$, $f_i^j \in End(R,+)$ will denote the map which is the sum of all possible words in σ, δ built with i letters σ and $j - i$ letters δ (e.g., $f_n^n = \sigma^n$ and $f_0^n = \delta^n, n \in \mathbb{N}$). The next lemma appears in [11, Lemma 4.1].

Lemma 2.1. *For any $n \in \mathbb{N}$ and $r \in R$ we have $x^n r = \sum\limits_{i=0}^{n} f_i^n(r) x^i$ in the ring*

$R[x;\sigma,\delta]$.

Lemma 2.2. *[5, Lemma 5]. Let R be an (σ,δ)-skew Armendariz ring. If $e^2 = e \in R[x;\sigma,\delta]$ where $e = e_0 + e_1x + e_2x^2 + \cdots + e_nx^n$, then $e = e_0$.*

Lemma 2.3. *Let R be a ring, σ an endomorphism and δ be a σ-derivation of R. Then $\sigma(Re) \subseteq Re$ implies $\delta(Re) \subseteq Re$ for all $e \in \mathcal{B}(R)$.*

Proof. Let $e \in \mathcal{B}(R)$ and $r \in R$. Then $\delta(re) = \delta(ere) = \sigma(er)\delta(e) + \delta(er)e = \sigma(ere)\delta(e) + \delta(er)e = se\delta(e) + \delta(er)e$, for some $s \in R$, but $e \in \mathcal{B}(R)$, then $e\delta(e) = e\delta(e)e$, so $\delta(re) = (se\delta(e) + \delta(er))e$. Therefore $\delta(Re) \subseteq Re$. □

Lemma 2.4. *Let R be a ring, σ an endomorphism of R and δ be a σ-derivation of R. If R is (σ,δ)-compatible. Then for $a, b \in R$, $ab = 0 \Rightarrow af_i^j(b) = 0$ for all $j \geq i \geq 0$.*

Proof. If $ab = 0$, then $a\sigma^i(b) = a\delta^j(b) = 0$ for all $i \geq 0$ and $j \geq 0$, because R is (σ,δ)-compatible. Then $af_i^j(b) = 0$ for all i, j. □

Lemma 2.5. *Let R be a ring, σ an endomorphism of R and δ be a σ-derivation of R. If R is σ-rigid then R is (σ,δ)-skew Armendariz.*

Proof. If R is σ-rigid then R is (σ,δ)-compatible by [4, Lemma 2.2]. Let $f = \sum_{i=0}^{n} a_ix^i$, $g = \sum_{j=0}^{m} b_jx^j \in R[x;\sigma,\delta]$ such that $fg = 0$, then $a_ib_j = 0$ for all i, j, by [7, Proposition 6]. So $a_if_\ell^j(b_j) = 0$, for all $0 \leq \ell \leq i \leq n$, $0 \leq j \leq m$, by Lemma 2.4. Hence $a_ix^ib_jx^j = \sum_{\ell=0}^{i} a_if_\ell^j(b_j)x^{\ell+j} = 0$. Therefore R is (σ,δ)-skew Armendariz. □

The next example illustrates that there exists a ring R and an automorphism σ of R such that Re is σ-stable for all $e \in \mathcal{S}_\ell(R)$, but R is not σ-rigid.

Example 2.6. *[8, Example 1]. Consider the ring*

$$R = \left\{ \begin{pmatrix} a & t \\ 0 & a \end{pmatrix} |a \in \mathbb{Z} \,, t \in \mathbb{Q} \right\},$$

where $\mathbb{Z}$ and $\mathbb{Q}$ are the set of all integers and all rational numbers, respectively. The ring R is commutative, let $\sigma\colon R \to R$ be an automorphism defined by $\sigma\left(\begin{pmatrix} a & t \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & t/2 \\ 0 & a \end{pmatrix}$.

(1) *R is not σ-rigid.*

$\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \sigma\left(\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}\right) = 0$, *but* $\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \neq 0$, *if* $t \neq 0$.

(2) *$\sigma(Re) \subseteq Re$ for all $e \in \mathcal{S}_\ell(R)$. R has only two idempotents:*

$e_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ *end* $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, *let* $r = \begin{pmatrix} a & t \\ 0 & a \end{pmatrix} \in R$, *we have* $\sigma(re_0) \in Re_0$ *and* $\sigma(re_1) \in Re_1$.

Also we have an example of an endomorphism σ of a ring R such that Re is σ-stable for all $e \in \mathcal{S}_\ell(R)$ and R is not σ-compatible.

Example 2.7. *Let $\mathbb{K}$ be a field and $R = \mathbb{K}[t]$ a polynomial ring over $\mathbb{K}$ with the endomorphism σ given by $\sigma(f(t)) = f(0)$ for all $f(t) \in R$.*
(1) R is not σ-compatible (so not σ-rigid). Take $f = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$ and $g = b_1 t + b_2 t^2 + \cdots + b_m t^m$, since $g(0) = 0$ so, $f\sigma(g) = 0$, but $fg \neq 0$.
(2) R has only two idempotents 0 and 1 so Re is σ-stable for all $e \in \mathcal{S}_\ell(R)$.

There is an example of a ring R and an endomorphism σ of R such that R is σ-skew Armendariz and R is not σ-compatible.

Example 2.8. *Consider a ring of polynomials over $\mathbb{Z}_2$, $R = \mathbb{Z}_2[x]$. Let $\sigma\colon R \to R$ be an endomorphism defined by $\sigma(f(x)) = f(0)$. Then:*
(i) R is not σ-compatible. Let $f = \overline{1} + x$, $g = x \in R$, we have $fg = (\overline{1} + x)x \neq 0$, however $f\sigma(g) = (\overline{1} + x)\sigma(x) = 0$.
(ii) R is σ-skew Armendariz [8, Example 5].

In the next example, $S = R/I$ is a ring and $\overline{\sigma}$ an endomorphism of S such that S is $\overline{\sigma}$-compatible and not $\overline{\sigma}$-skew Armendariz.

Example 2.9. *Let $\mathbb{Z}$ be the ring of integers and $\mathbb{Z}_2$ be the ring of integers modulo 4. Consider the ring*

$$R = \left\{ \begin{pmatrix} a & \overline{b} \\ 0 & a \end{pmatrix} | a \in \mathbb{Z}\,, \overline{b} \in \mathbb{Z}_4 \right\}.$$

Let $\sigma\colon R \to R$ be an endomorphism defined by $\sigma\left(\begin{pmatrix} a & \overline{b} \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & -\overline{b} \\ 0 & a \end{pmatrix}$. Take the ideal $I = \left\{ \begin{pmatrix} a & \overline{0} \\ 0 & a \end{pmatrix} | a \in 4\mathbb{Z} \right\}$ of R. Consider the factor ring

$$R/I \cong \left\{ \begin{pmatrix} \overline{a} & \overline{b} \\ 0 & \overline{a} \end{pmatrix} | \overline{a}, \overline{b} \in 4\mathbb{Z} \right\}.$$

(1) R/I is not $\overline{\sigma}$-skew Armendariz. In fact, $\left(\begin{pmatrix} \overline{2} & \overline{0} \\ 0 & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{2} & \overline{1} \\ 0 & \overline{2} \end{pmatrix} x\right)^2 = 0 \in (R/I)[x;\overline{\sigma}]$, but $\begin{pmatrix} \overline{2} & \overline{1} \\ 0 & \overline{2} \end{pmatrix} \overline{\sigma} \begin{pmatrix} \overline{2} & \overline{0} \\ 0 & \overline{2} \end{pmatrix} \neq 0$.
(2) R/I is $\overline{\sigma}$-compatible. Let $A = \begin{pmatrix} \overline{a} & \overline{b} \\ 0 & \overline{a} \end{pmatrix}$, $B = \begin{pmatrix} \overline{a'} & \overline{b'} \\ 0 & \overline{a'} \end{pmatrix} \in R/I$. If $AB = 0$ then $\overline{aa'} = 0$ and $\overline{ab'} = \overline{ba'} = 0$, so that $A\overline{\sigma}(B) = 0$. The same for the converse. Therefore R/I is $\overline{\sigma}$-compatible.

3 Ore extensions over quasi-Baer rings

It was proved in [1, Theorem 1.2], that if R is a quasi-Baer ring and σ an automorphism of R then $R[x;\sigma]$ is a quasi-Baer ring. The following example shows that " σ is an automorphism " is not a superfluous condition in Proposition 3.2.

Example 3.1. *[6, Example 2.8]. There is an example of a quasi-Baer ring R and an endomorphism σ of R such that $R[x;\sigma]$ is not a quasi-Baer ring. In fact, let $R=\mathbb{K}[t]$ be the polynomial ring over a field $\mathbb{K}$ and σ be the endomorphism given by $\sigma(f(t))=f(0)$. Then the ring $R[x;\sigma]$ is not a quasi-Baer ring.*

Proposition 3.2. *Let R be a ring, σ an automorphism and δ be a σ-derivation of R. Suppose that Re is (σ,δ)-stable for all $e\in\mathcal{S}_\ell(R)$. If R is quasi-Baer then the Ore extension $R[x;\sigma,\delta]$ is quasi-Baer.*

Proof. Let $S=R[x;\sigma,\delta]$ and I be an ideal of S. We claim that $r_S(I)=eS$, for some idempotent $e\in R$. We can suppose that $I\neq 0$, we set

$I_0=\{0\}\cup\{a\in R\mid \exists\, a_0,a_1,\cdots,a_{n-1}\in R \text{ such that } ax^n+\sum_{i=0}^{n-1}a_ix^i\in I, n\in\mathbb{N}\}$.

It is clear that I_0 is a nonzero left ideal of R. Given $a\in I_0$ and $r\in R$, there is an element in I of the form $ax^n+\sum_{i=0}^{n-1}a_ix^i$. Multiplying on the right by $\sigma^{-n}(r)$ gives an element of the form $arx^n+\sum_{i=0}^{n-1}b_ix^i$, for some elements $b_0,b_1,\cdots,b_{n-1}\in R$, and so $ar\in I_0$, thus I_0 is a two-sided ideal. So there exists an idempotent $e\in R$ such that $r_R(I_0)=eR$. We have $eS\subseteq r_S(I)$. To see this, let $0\neq f(x)=\sum_{k=0}^{n}a_kx^k\in I$, then $f(x)e=\sum_{k=0}^{n}(\sum_{i=k}^{n}a_kf_k^i(e))x^k$, where f_k^i are sums of all possible words in σ,δ built with k letters σ and $i-k$ letters δ. Re is f_k^i-stable $(0\leq k\leq i)$, so there exists $u_k^i\in R$ such that $f_k^i(e)=u_k^ie$ $(0\leq k\leq i)$. Therefore $f(x)e=\sum_{k=0}^{n}(\sum_{i=k}^{n}a_ku_k^i)ex^k$, if we set $\alpha_k=\sum_{i=k}^{n}a_ku_k^ie$, then $f(x)e=\sum_{k=0}^{n}\alpha_kx^k$. If $\alpha_n\neq 0$, then $\alpha_n\in I_0$ and so, $\alpha_ne=\alpha_n=0$ (because $r_R(I_0)=eR$). Contradiction, hence $\alpha_n=0$. Now suppose that $\alpha_j=0$ for $j=n,n-1,\cdots,k+1$ with $k\in\mathbb{N}$. But $f(x)e=\alpha_kx^k+\sum_{\ell=0}^{k-1}\alpha_\ell x^\ell$, with the same manner as above we have $\alpha_k=0$. So we can get $\alpha_n=\alpha_{n-1}=\cdots=\alpha_0=0$. Consequently $eS\subseteq r_S(I)$.

Conversely, we can claim that $r_S(I)\subseteq eS$. Let $0\neq f(x)=\sum_{k=0}^{n}a_kx^k\in I$ and $\lambda(x)=\sum_{j=0}^{m}b_jx^j\in S$, such that $f(x)\lambda(x)=0$, we shall show that $\lambda(x)=$

$\sigma^{-n}(e)\lambda(x)$. If we set $\xi(x) = \lambda(x) - \sigma^{-n}(e)\lambda(x) = \sum_{j=0}^{m}(b_j - \sigma^{-n}(e)b_j)x^j$, we have $f(x)\xi(x) = (\sum_{i=0}^{n} a_i x^i)(\sum_{j=0}^{m}(b_j - \sigma^{-n}(e)b_j)x^j) = a_n\sigma^n(b_m - \sigma^{-n}(e)b_m)x^{n+m} + Q = 0$, where Q is a polynomial with $deg(Q) < n+m$. Thus $a_n\sigma^n(b_m - \sigma^{-n}(e)b_m) = 0$, since $a_n \neq 0$, then $a_n \in I_0$. Hence $\sigma^n(b_m - \sigma^{-n}(e)b_m) \in r_R(I_0) = eR$. So $\sigma^n(b_m - \sigma^{-n}(e)b_m) = e\sigma^n(b_m - \sigma^{-n}(e)b_m)$, then $b_m - \sigma^{-n}(e)b_m = \sigma^{-n}(e)(b_m - \sigma^{-n}(e)b_m) = 0)$ (because $\sigma^{-n}(e)$ is idempotent), hence $b_m - \sigma^{-n}(e)b_m = 0$. Now, suppose that $b_j - \sigma^{-n}(e)b_j = 0$ for $j = m, m-1, \cdots, k+1$ with $k \in \mathbb{N}$ and showing that $b_k - \sigma^{-n}(e)b_k = 0$. Effectively, $f(x)\xi(x) = a_n\sigma^n(b_k - \sigma^{-n}(e)b_k)x^{n+k} + Q' = 0$, where Q' is a polynomial with $deg(Q') < n+k$, then $a_n\sigma^n(b_k - \sigma^{-n}(e)b_k) = 0$, with the same manner as below, we obtain $b_k - \sigma^{-n}(e)b_k = 0$. Therefore $b_j - \sigma^{-n}(e)b_j = 0$ for all $0 \leq j \leq m$, then $\xi(x) = 0$. But $\lambda(x) = \sigma^n(e)\lambda(x)$ or $\sigma^n(e) = ue$ for some $u \in R$, but e is left semicentral then $\lambda(x) = eue\lambda(x)$. Hence $r_S(I) \subseteq eS$. So $R[x;\sigma,\delta]$ is a quasi-Baer ring. □

In Example 2.7, Re is (σ,δ)-stable for all $e \in \mathcal{S}_\ell(R)$ but R is not (σ,δ)-compatible. Thus, Proposition 3.2 is not a consequence of [4, Corollary 2.8].

There is a quasi-Baer ring R, σ an automorphism of R and δ a σ-derivation of R such that Re is (σ,δ)-stable for all $e \in \mathcal{S}_\ell(R)$.

Example 3.3. *Consider the ring* $R = \begin{pmatrix}\mathbb{Z} & \mathbb{Z}\\ 0 & \mathbb{Z}\end{pmatrix}$, *where* $\mathbb{Z}$ *is the set of all integers numbers. By [2, Example 1.3(ii)],* R *is a quasi-Baer ring. Define* $\sigma\colon R \to R$ *and* $\delta\colon R \to R$ *by*

$$\sigma\left(\begin{pmatrix}a & b\\ 0 & c\end{pmatrix}\right) = \begin{pmatrix}a & -b\\ 0 & c\end{pmatrix}, \quad \delta\left(\begin{pmatrix}a & b\\ 0 & c\end{pmatrix}\right) = \begin{pmatrix}0 & 2b\\ 0 & 0\end{pmatrix} \text{ for all } a,b,c \in \mathbb{Z}.$$

Clearly, σ *is an automorphism of* R *and* δ *is a* σ*-derivation. The nonzero idempotents of* R *are of the form*

$$e_0 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ e_1 = \begin{pmatrix}1 & t\\ 0 & 0\end{pmatrix} \text{ and } e_2 = \begin{pmatrix}0 & t\\ 0 & 1\end{pmatrix},$$

where $t \in \mathbb{Z}$. e_2 *is right semicentral not left semicentral and* e_1 *is left semicentral not right semicentral, so the only left semicentral nonzero idempotents of* R *are* e_0 *and* e_1. Re_0 *is* (σ,δ)*-stable. Let* $r = \begin{pmatrix}x & y\\ 0 & z\end{pmatrix} \in R$, *since* $\sigma(re_1) = \begin{pmatrix}x & -xt\\ 0 & 0\end{pmatrix} \in \begin{pmatrix}\mathbb{Z} & \mathbb{Z}\\ 0 & \mathbb{Z}\end{pmatrix}\begin{pmatrix}1 & t\\ 0 & 0\end{pmatrix}$, *then* Re_1 *is* σ*-stable, also* Re_1 *is* δ*-stable. Therefore* Re *is* (σ,δ)*-stable for all* $e \in \mathcal{S}_\ell(R)$.

Example 3.4. *Consider the ring* $S = \begin{pmatrix} D & D \oplus D \\ 0 & D \end{pmatrix}$, *where* D *is a simple domain which is not a division ring. By [3, Example 4.11],* R *is a quasi-Baer ring and has nonzero idempotents of the form*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & (b,d) \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & (b,d) \\ 0 & 1 \end{pmatrix},$$

where $b, d \in D$, *with* σ *and* δ *as in Example 3.3,* Re *is* (σ, δ)*-stable for all* $e \in \mathcal{S}_\ell(R)$.

Corollary 3.5. *Let* R *be an abelian or a semiprime ring,* σ *an automorphism and* δ *be a* σ*-derivation of* R, *such that* $\sigma(Re) \subseteq Re$ *for all* $e \in \mathcal{B}(R)$. *If* R *is quasi-Baer then* $R[x; \sigma, \delta]$ *is quasi-Baer.*

Proof. By Lemma 2.3 and Proposition 3.2. □

In the remainder of this section we focus on the converse of Proposition 3.2. We begin with the next example which shows that there exists a ring R and a derivation δ of R such that $R[x; \delta]$ is quasi-Baer but R is not quasi-Baer.

Example 3.6. *[1, Example 1.6]. There is a ring* R *and a derivation* δ *of* R *such that* $R[x; \delta]$ *is a Baer ring. But* R *is not quasi-Baer. Let* $R = \mathbb{Z}_2[t]/(t^2)$ *with the derivation* δ *such that* $\delta(\bar{t}) = 1$ *where* $\bar{t} = t + (t^2)$ *in* R *and* $\mathbb{Z}_2[t]$ *is the polynomial ring over the field* $\mathbb{Z}_2$ *of two elements. Consider the Ore extension* $R[x; \delta]$. *If we set* $e_{11} = \bar{t}x, e_{12} = \bar{t}, e_{21} = \bar{t}x^2 + x$ *and* $e_{22} = 1 + \bar{t}x$ *in* $R[x; \delta]$, *then they form a system of matrix units in* $R[x; \delta]$. *Now the centralizer of these matrix units in* $R[x; \delta]$ *is* $\mathbb{Z}_2[x^2]$. *Therefore* $R[x; \delta] \cong M_2(\mathbb{Z}_2[x^2]) \cong M_2(\mathbb{Z}_2)[y]$, *where* $M_2(\mathbb{Z}_2)[y]$ *is the polynomial ring over* $M_2(\mathbb{Z}_2)$. *So the ring* $R[x; \delta]$ *is a Baer ring, but* R *is not quasi-Baer.*

Proposition 3.7. *Let* R *be an* (σ, δ)*-skew Armendariz ring. If* $R[x; \sigma, \delta]$ *is quasi-Baer then* R *is quasi-Baer.*

Proof. Let I be an ideal of R and $S = R[x; \sigma, \delta]$, then since S is quasi-Baer, there exists an idempotent $e \in S$ such that $r_S(IS) = eS$ with $e = e_0 + e_1x + \cdots + e_nx^n$ $(n \in \mathbb{N})$. By Lemma 2.2, we have $e_0 \in r_R(I)$. Thus $e_0R \subseteq r_R(I)$.

Conversely, let $a \in r_R(I)$ then $a \in r_S(IS) \cap R = e_0S \cap R$, so $a = e_0f$ for some $f = f_0 + f_1x + \cdots + f_mx^m \in S$. Then $a = e_0f_0$ and so $a \in e_0R$. Therefore $r_R(I) \subseteq e_0R$. Consequently, R is a quasi-Baer ring. □

By Example 2.8, there is a ring R and σ an endomorphism of R such that R is σ-skew Armendariz and R is not σ-compatible. So that, Proposition 3.7 is not a consequence of [4, Corollary 2.8]. By the next result, we see that Proposition 3.7 is a partial generalization of [7, Corollary 12].

Corollary 3.8. *Let R be an σ-rigid ring. If $R[x;\sigma,\delta]$ is quasi-Baer then R is quasi-Baer.*

Proof. It follows from Lemma 2.5 and Proposition 3.7. □

One might expect the converse of Proposition 3.2 to hold when R is a (σ,δ)-skew Armendariz ring. However [8, Example 5] and [6, Example 2.8], shows that this converse does not hold in general.

Example 3.9. *We consider a commutative polynomial ring over $\mathbb{Z}_2$. $R = \mathbb{Z}_2[x]$, let $\sigma\colon R \to R$ be an endomorphism defined by $\sigma(f(x)) = f(0)$. By [6, Example 2.8], $R[x;\sigma]$ is not quasi-Baer and R is quasi-Baer. But, by [8, Example 5], R is σ-skew Armendariz. Note that R has only two idempotents 0 and 1, so $\sigma(Re) \subseteq Re$ for all $e \in \mathcal{S}_\ell(R)$. Thus " σ is an automorphism " is not a superfluous condition in the next theorem.*

Theorem 3.10. *Let R be a (σ,δ)-skew Armedariz ring with σ an automorphism such that Re is (σ,δ)-stable for all $e \in \mathcal{S}_\ell(R)$. Then R is a quasi-Baer ring if and only if $R[x;\sigma,\delta]$ is a quasi-Baer ring.*

Proof. It follows immediately from Proposition 3.2 and Proposition 3.7. □

Example 3.11. *Let $R = \mathbb{C}$ where $\mathbb{C}$ is the field of complex numbers. Then R is a Baer (so quasi-Baer) reduced ring. Define $\sigma\colon R \to R$ and $\delta\colon R \to R$ by $\sigma(z) = \overline{z}$ and $\delta(z) = z - \overline{z}$, where $\overline{z}$ is the conjugate of z. σ is an automorphism of R and δ is a σ-derivation. R has only two idempotents 0 and 1, so we have the stability indicated in Theorem 3.10.*

We claim that R is a (σ,δ)-skew Armendariz ring. Consider $R[x;\sigma,\delta]$. Let $p = a_0 + a_1x + \cdots + a_nx^n$ and $q = b_0 + b_1x + \cdots + b_mx^m \in R[x;\sigma,\delta]$. Assume that $pq = 0$. Since R is σ-rigid, we have $a_ib_j = 0$ for all $0 \le i \le n$ and $0 \le j \le m$, by [7, Proposition 6]. thus $a_ix^ib_jx^j = 0$ for all $0 \le i \le n$ and $0 \le j \le m$, because $R[x;\sigma,\delta]$ is reduced, by [10, Theorem 3.3].

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References

[1] G.F. Birkenmeier, J.Y. Kim, J.K. Park, *Polynomial extensions of Baer and quasi-Baer rings*, J. Pure appl. Algebra 159: (2001) 25-42.

[2] G.F. Birkenmeier, J.Y. Kim, J.K. Park, *Principally quasi-Baer rings*, Comm. Algebra 29(2): (2001) 639-660.

[3] G.F. Birkenmeier, B.J. Müller, S.T. Rizvi, *Modules in which every fully invariant submodule is issential in a direct summand*, Comm. Algebra 30(3): (2002) 1395-1415.

[4] E. Hashemi, A. Moussavi, *Polynomial extensions of quasi-Baer rings* , Acta math. Hungar. 107(3): (2005) 207-224.

[5] E. Hashemi, A. Moussavi, *On* (σ, δ)*-skew Armendariz rings* , J. Korean Math. soc. 42(2): (2005) 353-363.

[6] J. Han, Y. Hirano, H. Kim, *Some results on skew polynomial rings over a reduced ring*, In: G.F. Birkenmeier, J.K. Park, Y.S. Park (Eds), The international symposium on ring theory, In: Trends in math., Birkhäuser Boston (2001).

[7] C.Y. Hong, N.K. Kim, T.K. Kwak, *Ore extensions of Baer and p.p.-rings*, J. Pure and Appl. Algebra 151(3): (2000) 215-226.

[8] C.Y. Hong, N.K. Kim, T.K. Kwak, *On Skew Armendariz Rings*, Comm. Algebra 31(1): (2003) 103-122.

[9] I. Kaplansky, *Rings of operators*, Math.lecture Notes series , Benjamin, New York (1965).

[10] J. Krempa, *Some examples of reduced rings*, Algebra Colloq. 3(4): (1996) 289-300.

[11] T.Y. Lam, A. Leroy, J. Matczuk, *Primeness, semiprimeness and the prime radical of Ore extensions*, Comm. Algebra 25(8): (1997) 2459-2506.