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A P-ADIC GENERALIZED BOREL'S LEMMA

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Abstract

In this paper, by using p-adic Nevanlinna theory, we prove a generalized Borel's lemma in the p-adic case.

1 Introduction

Let us start by recalling Borel's lemma in the complex case:

Theorem 1.1. *Let $f_1, \ldots, f_n, n \geq 3$ be non-zero holomorphic functions on $\mathbb{C}$ such that*

$$f_1 + \cdots + f_n = 0.$$

Then the function $\{f_1, \ldots, f_{n-1}\}$ are linearly dependent.

It is well - known that Borel's lemma plays an important role in the study of hyperbolic spaces. For different purposes some generalizations of the lemma are given. We mention here a result of Y.T. Siu and S.K. Yeung.

Theorem 1.2 ([13]). *Let $g_j(x_0, \ldots, x_n)$ be a homogeneous polynomial of degree δ_j for $0 \leq j \leq n$. Suppose there exists a holomorphic map $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ so that its image lies in*

$$\sum_{j=0}^{n} x_j^{k-\delta_j} g_j(x_0, \ldots, x_n) = 0,$$

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and $k > (n+1)(n-1) + \sum_{j=0}^{n} \delta_j$. *Then there is a nontrivial linear relation among* $x_1^{k-\delta_1} g_1(x_0, \ldots, x_n), \ldots, x_n^{k-\delta_n} g_n(x_0, \ldots, x_n)$ *on the image of* f.

In [12], by using p-adic Nevanlinna-Cartan's theorem, Nguyen Thanh Quang and Phan Duc Tuan proved a p-adic version of Siu-Yeung's lemma as the follows.

Theorem 1.3. *Let* $g_j(x_0, \ldots, x_n)$ *be a homogeneous polynomial of degree* δ_j *for* $0 \leq j \leq n$. *Suppose there exists a holomorphic map* $f : \mathbb{C}_p \to \mathbb{P}^n(\mathbb{C}_p)$ *so that its image lies in*

$$\sum_{j=0}^{n} x_j^{k-\delta_j} g_j(x_0, \ldots, x_n) = 0,$$

and

$$k \geq (n+1)(n-1) + \sum_{j=0}^{n} \delta_j.$$

Then the following functions are linearly dependent on $\mathbb{C}_p$ *if they are have no common zeros:*

$$f_1^{k-\delta_1} g_1(f_0, \ldots, f_n), \ldots, f_n^{k-\delta_n} g_n(f_0, \ldots, f_n).$$

In this paper, by using p-adic Nevanlinna theory, we prove an analog of a generalized *abc*-conjecture for p-adic entire functions and we then apply the result to prove the hypothesis that the functions

$$f_1^{k-\delta_1} g_1(f_0, \ldots, f_n), \ldots, f_n^{k-\delta_n} g_n(f_0, \ldots, f_n)$$

have no common zeros in Theorem 1.3 is not necessary.

2 An analog of a generalized abc-conjecture for p-adic entire functions

Let p be a prime number, $\mathbb{Q}_p$ the field of p-adic number, and let $\mathbb{C}_p$ be the p-adic completion of the algebraic closure of $\mathbb{Q}_p$. The absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p = p^{-1}$.

Let $a \in \mathbb{C}_p$ and f is a p-adic meromorphic function, we write f in the form:

$$f = (x-a)^l \frac{g}{h}$$

Where g, h are ertire functions and $g(a)h(a) \neq 0$, then l is called the *order of* f *at* a and is denoted by μ_f^a, we also denote $\mu_{f,k}^a = \min(k, \mu_f^a)$. We have the following easily proved properties of μ_f^a.

Lemma 2.1. *Let f, g be two meromorphic functions and $a \in \mathbb{C}_p$, we have*
a) $\mu^a_{f+g} \geq \min(\mu^a_f, \mu^a_g)$,
b) $\mu^a_{fg} = \mu^a_f + \mu^a_g$,
c) $\mu^a_{\frac{f}{g}} = \mu^a_f - \mu^a_g$,
d) $\mu^a_{f^{(k)}} \geq \mu^a_f - k$.

Let f be a nonconstant p-adic analytic function on $\mathbb{C}_p$,

$$f = \sum_{n=0}^{\infty} a_n z^n (a_n \in \mathbb{C}_p)$$

is well-defined whenever

$$|a_n z^n|_p \to 0.$$

Define the maximum term

$$\mu(r, f) = \max_{n \geq 0} |a_n| r^n.$$

Let f is a meromorphic function, we can uniquely extend μ to meromorphic function $f = \frac{g}{h}$ by defining

$$\mu(r, f) = \frac{\mu(r, g)}{\mu(r, h)} (0 \leq r < \infty).$$

Define the *compensation function* by

$$m(r, f) = \max\{0, \log \mu(r, f)\}.$$

Define the *counting function* by

$$N\left(r, \frac{1}{f}\right) = \mu^0_f \log r + \sum_{0 < a \leq r, f(a)=0} \mu^a_f \log |\frac{r}{a}|,$$

and

$$\overline{N}\left(r, \frac{1}{f}\right) = \mu^0_{f,1} \log r + \sum_{0 < a \leq r, f(a)=0} \mu^a_{f,1} \log |\frac{r}{a}|.$$

As usual, we define the *characteristic function by*

$$T(r, f) = m(r, f) + N(r, f),$$

where

$$N(r, f) = N(r, \frac{1}{h}).$$

We have *Jensen Formular* (see [3])

$$N(r, \frac{1}{f}) - N(r, f) = \log \mu(r, f) + 0(1).$$

Now we give an analog of a generalized *abc*-conjecture for p-adic entire functions.

Theorem 2.1. *Let $f_0, ..., f_{n+1}$ be $(n+2)$ entire functions have no common zeros and $g_0, ..., g_{n+1}$ be $(n+2)$ entire functions such that $f_0g_0, \dots, f_ng_n$ be linearly independent over $\mathbb{C}_p$, and*

$$f_0g_0 + \cdots + f_ng_n = f_{n+1}g_{n+1}$$

Then

$$\max_{0 \le j \le n+1} T(r, f_jg_j) \le n \sum_{j=0}^{n+1} \overline{N}(r, \frac{1}{f_j}) + \sum_{j=0}^{n+1} N(r, \frac{1}{g_j}) - \frac{n(n+1)}{2} \log r + 0(1).$$

Proof. Since $f_0g_0, ..., f_ng_n$ are linearly independent, then the Wronskian W of $f_0g_0, ..., f_ng_n$ does not vanish. We set

$$P = \frac{W(f_0g_0, ..., f_ng_n)}{f_0g_0....f_ng_n},$$

$$Q = \frac{f_0g_0....f_{n+1}g_{n+1}}{W(f_0g_0, ..., f_ng_n)}.$$

Hence we have

$$f_{n+1}g_{n+1} = PQ.$$

By Jensen formular, we have

$$T(r, f_{n+1}g_{n+1}) = N(r, \frac{1}{f_{n+1}g_{n+1}}) + 0(1) \le N(r, \frac{1}{Q}) + N(r, \frac{1}{P}) + 0(1). \quad (1)$$

We have

$$N(r, \frac{1}{Q}) = N\left(r, \frac{1}{\frac{f_0 \cdots f_{n+1}}{W(f_0g_0, \dots, f_ng_n)}}\right) + \sum_{j=0}^{n+1} N(r, \frac{1}{g_j}) + 0(1). \quad (2)$$

Suppose that α is a zero of $f_0f_1 \cdots f_{n+1}$, by the hypothesis there exists $\nu, 0 \le \nu \le n+1$ such that $f_\nu \ne 0$. By the hypothesis $f_0g_0 + \cdots + f_ng_n = f_{n+1}g_{n+1}$ we have

$$\begin{aligned} \mu^\alpha_{\frac{f_0 \cdots f_{n+1}}{W(f_0g_0, \dots, f_ng_n)}} &= \mu^\alpha_{\frac{f_0 \cdots f_{\nu-1} f_{\nu+1} \cdots f_{n+1}}{W(f_0g_0, \dots, f_{\nu-1}g_{\nu-1}, f_{\nu+1}g_{\nu+1}, \dots, f_{n+1}g_{n+1})}} \\ &= \sum_{j=0}^{n+1} \mu^\alpha_{f_j} - \mu^\alpha_{W(f_0g_0, \dots, f_{\nu-1}g_{\nu-1}, f_{\nu+1}g_{\nu+1}, \dots, f_{n+1}g_{n+1})} \end{aligned}$$

$W(f_0g_0,...,f_{\nu-1}g_{\nu-1},f_{\nu+1}g_{\nu+1},...,f_{n+1}g_{n+1})$ is the sum of follow terms

$$\delta f_{\alpha_0}g_{\alpha_0}(f_{\alpha_1}g_{\alpha_1})'\cdots(f_{\alpha_n}g_{\alpha_n})^{(n)},$$

Where $\alpha_i\in\{0,...n+1\}\backslash\{\nu\},\delta=\pm1$. By using Lemma 2.1 we have

$$\begin{aligned}
&\mu^{\alpha}_{f_{\alpha_0}g_{\alpha_0}(f_{\alpha_1}g_{\alpha_1})'\cdots(f_{\alpha_n}g_{\alpha_n})^{(n)}}\\
\geq\ & \sum_{0\leq j\leq n,f_{\alpha_j}(\alpha)=0}\mu^{\alpha}_{f_{\alpha_j}g_{\alpha_j}}-n(\sum_{0\leq j\leq n,f_{\alpha_j}(a)=0}1)\\
=\ & \sum_{0\leq j\leq n,f_{\alpha_j}(\alpha)=0}\mu^{\alpha}_{f_{\alpha_j}}+\sum_{0\leq j\leq n,f_{\alpha_j}(\alpha)=0}\mu^{\alpha}_{g_{\alpha_j}}-n(\sum_{0\leq j\leq n,f_{\alpha_j}(a)=0}1)\\
\geq\ & \sum_{0\leq j\leq n,f_{\alpha_j}(\alpha)=0}\mu^{\alpha}_{f_{\alpha_j}}-n(\sum_{0\leq j\leq n+1,f_{\alpha_j}(a)=0}1)\\
=\ & \sum_{j=0}^{n+1}\mu^{\alpha}_{f_j}-n(\sum_{0\leq j\leq n+1,f_j(a)=0}1).
\end{aligned}$$

By Lemma 2.1 we have

$$\mu^{\alpha}_{W(f_0g_0,...,f_{\nu-1}g_{\nu-1},f_{\nu+1}g_{\nu+1},...f_{n+1}g_{n+1})}\geq\sum_{j=0}^{n+1}\mu^{\alpha}_{f_{\alpha_j}}-n(\sum_{0\leq j\leq n+1,f_j(a)=0}1).$$

Hence

$$\mu^{\alpha}_{\frac{f_0\cdots f_{n+1}}{W(f_0g_0,\cdots,f_ng_n)}}\leq n(\sum_{0\leq j\leq n+1,f_j(a)=0}1).$$

By the definition of counting function, we have:

$$N\left(r,\frac{1}{\frac{f_0\cdots f_{n+1}}{W(f_0g_0,...,f_ng_n)}}\right)\leq n\sum_{j=0}^{n+1}\overline{N}(r,\frac{1}{f_j}) \tag{3}$$

By Jensen formular we have

$$N(r,\frac{1}{P})\leq\log\mu(r,P)+0(1). \tag{4}$$

The value of P is clearly

$$\sum\pm\frac{f_{\alpha_0}g_{\alpha_0}}{f_{\alpha_0}g_{\alpha_O}}\cdot\frac{(f_{\alpha_1}g_{\alpha_1})'}{f_{\alpha_1}g_{\alpha_1}}\cdots\frac{(f_{\alpha_n}g_{\alpha_n})^{(n)}}{f_{\alpha_n}g_{\alpha_n}}$$

summed for the $(n+1)!$ permutations $(\alpha_0,...,\alpha_n)$ of $(0,...,n)$, the positive sign being taken for a positive permutation, the negative sign for a negative permutation. We have

$$\begin{aligned}\mu(r,P) &\leq \max \mu\left(r, \frac{(f_{\alpha_1}g_{\alpha_1})'}{f_{\alpha_1}g_{\alpha_1}}\cdots\frac{(f_{\alpha_n}g_{\alpha_n})^{(n)}}{f_{\alpha_n}g_{\alpha_n}}\right)\\ &= \max \mu\left(r, \frac{(f_{\alpha_1}g_{\alpha_1})'}{f_{\alpha_1}g_{\alpha_1}}\right)\cdots\mu\left(r, \frac{(f_{\alpha_n}g_{\alpha_n})^{(n)}}{f_{\alpha_n}g_{\alpha_n}}\right)\end{aligned}$$

By using the lemma of logarithmic derivative (see [2]) which states

$$\mu\left(r, \frac{f^{(i)}}{f}\right) \leq \frac{1}{r^i}$$

for a nonconstant meromorphic function f in C_p, we obtain

$$\mu(r,P) \leq \frac{1}{r}\cdot\frac{1}{r^2}\cdots\frac{1}{r^n} = r^{-\frac{n(n+1)}{2}}. \tag{5}$$

From (4) and (5) we have

$$N(r, \frac{1}{P}) \leq -\frac{n(n+1)}{2}\log r + 0(1). \tag{6}$$

From $(1), (2), (3)$ and (6) we have

$$T(r, f_{n+1}g_{n+1}) \leq n\sum_{j=0}^{n+1}\overline{N}(r, \frac{1}{f_j}) + \sum_{j=0}^{n+1}N(r, \frac{1}{g_j}) - \frac{n(n+1)}{2}\log r + 0(1).$$

By a similar argument applying to the functions $f_0g_0, ..., f_ng_n$, we have

$$\max_{0\leq j\leq n+1} T(r, f_jg_j) \leq n\sum_{j=0}^{n+1}\overline{N}(r, \frac{1}{f_j}) + \sum_{j=0}^{n+1}N(r, \frac{1}{g_j}) - \frac{n(n+1)}{2}\log r + 0(1).$$

Theorem 2.1 is proved. □

3 A P-adic generalized Borel's lemma

The following theorem is more general than Theorem 1.3.

Theorem 3.1. *Let $g_j(x_0, \ldots, x_n)$ be a homogeneous polynomial of degree δ_j for $0 \leq j \leq n$. Suppose there exists a holomorphic map $f : \mathbb{C}_p \to \mathbb{P}^n(\mathbb{C}_p)$ so that its image lies in*

$$\sum_{j=0}^{n} x_j^{k-\delta_j} g_j(x_0, \ldots, x_n) = 0,$$

and $k \geq (n+1)(n-1) + \sum_{j=0}^{n}\delta_j$. Then there is a nontrivial linear relation among $x_1^{k-\delta_1}g_1(x_0, \ldots, x_n), \ldots, x_n^{k-\delta_n}g_n(x_0, \ldots, x_n)$ on the image of f.

Proof. By the hypothesis we have

$$\sum_{j=0}^{n} f_j^{k-\delta_j} g_j(f_0,\ldots,f_n)=0,$$

Assume on the contrary that the functions

$$f_1^{k-\delta_1} g_1(f_0,\ldots,f_n),\ldots,f_n^{k-\delta_n} g_n(f_0,\ldots,f_n)$$

are linearly independent. By using Theorem 2.1 we have

$$\begin{aligned}
&\max\Big(T(r,f_0^{k-\delta_0}g_1(f_0,...,f_n)),...,T(r,f_n^{k-\delta_n}g_n(f_0,...,f_n))\Big)\\
\leq\ & (n-1)\sum_{j=0}^{n+1}\overline{N}(r,\frac{1}{f_j^{k-\delta_j}})+\sum_{j=0}^{n+1}N(r,\frac{1}{g_j(f_0,...,f_n)})-\frac{n(n-1)}{2}\log r+0(1)\\
=\ & (n-1)\sum_{j=0}^{n+1}\overline{N}(r,\frac{1}{f_j})+\sum_{j=0}^{n+1}N(r,\frac{1}{g_j(f_0,...,f_n)})-\frac{n(n-1)}{2}\log r+0(1)\\
\leq\ & (n-1)\sum_{j=0}^{n+1}N(r,\frac{1}{f_j})+\sum_{j=0}^{n+1}N(r,\frac{1}{g_j(f_0,...,f_n)})-\frac{n(n-1)}{2}\log r+0(1)
\end{aligned}$$

.

We set for simplicity

$$\max_{0\leq j\leq n}T(r,f_j)=T(r,f_{i_0}).$$

Then we have

$$\begin{aligned}
&T(r,f_{i_0}^{k-\delta_{i_0}}g_{i_0}(f_0,...,f_n))\\
\leq\ & (n-1)\sum_{j=0}^{n+1}N(r,\frac{1}{f_j})+\sum_{j=0}^{n+1}N(r,\frac{1}{g_j(f_0,...,f_n)})-\frac{n(n-1)}{2}\log r+0(1).
\end{aligned}$$

Hence

$$\begin{aligned}
&N(r,\frac{1}{f_{i_0}^{k-\delta_{i_0}}})+N(r,\frac{1}{g_{i_0}(f_0,...,f_n)})\\
\leq\ & (n-1)\sum_{j=0}^{n+1}N(r,\frac{1}{f_j})+\sum_{j=0}^{n+1}N(r,\frac{1}{g_j(f_0,...,f_n)})-\frac{n(n-1)}{2}\log r+0(1).
\end{aligned}$$

Thus

$$\begin{aligned}
&(k-\delta_{i_0})N(r,\tfrac{1}{f_{i_0}})\\
\leq\ & (n-1)(n+1)N(r,\tfrac{1}{f_{i_0}})+\sum_{0\leq j\leq n,j\neq i_0}^{n+1}N(r,\tfrac{1}{g_j(f_0,\ldots,f_n)})-\tfrac{n(n-1)}{2}\log r+0(1)\\
\leq\ & (n-1)(n+1)N(r,\tfrac{1}{f_{i_0}})+\sum_{0\leq j\leq n,j\neq i_0}^{n+1}\delta_jN(r,\tfrac{1}{f_{i_0}})-\tfrac{n(n-1)}{2}\log r+0(1).
\end{aligned}$$

So

$$(k - \sum_{j=0}^{n} \delta_j - (n-1)(n+1))N(r, \frac{1}{f_{i_0}}) \leq -\frac{(n-1)n}{2} \log r + 0(1).$$

By the hypothesis $k \geq \sum_{j=0}^{q} \delta_j - (n-1)(n+1)$ we have a contradiction when $r \to +\infty$. $\square$

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