East-West J. of Mathematics: Vol. 8, No 2 (2006) pp. 163-180

CLASSES OF EXTENDING MODULES ASSOCIATED WITH A TORSION THEORY

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Abstract

A module M over an associative ring R is said to be *extending* if every submodule of M is essential in some direct summand of M. In this paper we consider τ*-extending* modules, an analogue of extending that corresponds to a torsion theory τ on the modules over R. We present some fundamental properties of τ-extending modules, relate them to extending modules, and give some results regarding when direct sums of τ-extending modules are τ-extending. Examples are provided to illustrate our results. For τ_G the Goldie torsion theory and τ a larger torsion theory, the τ_G-extending modules coincide with the τ-extending modules.

1 Introduction

For R an associative ring with identity, the study of extending modules and their various generalizations originated with von Neumann's work in the 1930s on the development of "continuous geometry" and "continuous" von Neumann regular rings. He uses the term "geometry" to mean lattices; the rings are said to be "continuous " provided the lattice of principal right ideals is *upper and lower continuous* [31], [32]; that is,
for all $\{a\}, \{b_i\}_{i\in I} \subseteq R$ and an index set I, **(uc)** and **(lc)** hold:
(uc) $aR \cap (\sum_{i\in I} b_iR) = \sum_{i\in I}(aR \cap b_iR)$, and

Key words: closed submodule, extending module, torsion theory.
2000 AMS Mathematics Subject Classification: 16S90.
The research was supported in part by TUBITAK-NATO-B2(TURKEY) sicience Fellowship Programme during her sabbatical year in the University of Nebraska-Lincoln 2004/2005.

(lc) $aR + (\cap_{i\in I} b_i R) = \cap_{i\in I}(aR + b_i R)$.
(Originally von Neumann worked with left ideals, but we use right ideals and modules in this paper.) Later Utumi [30], Jeremy [20], Takeuchi [29] and Mohamed-Bouhy [24] generalized *continuous* to non-regular rings and modules. There is a good account of this history in Mohamed and Muller [23] and Dung, Huynh, Smith and Wisbauer [7].

We pause for some conventions and definitions before giving more history. Throughout; R and all rings are associative with identity and all modules are unitary right modules. We let Mod-R denote the category of unitary right R-modules.

Definitions 1.1. Let R be a ring and M be an R-module. A nonzero submodule N of M is an *essential submodule* of M, written $N \leq_e M$, if $N \cap K \neq (0)$ for every nonzero submodule K of M. For N a submodule of M, a *closure* of N (in M) is a submodule K of M that is maximal in the collection of submodules H of M that contain N as an essential submodule. We say K is *closed* (in M), written $K \leq_c M$, if K has no proper essential extension in M. The module M is *extending* if every closed submodule of M is a direct summand, or equivalently, if every submodule of M is essential in a direct summand of M.

Remarks. (1) Because of the disparate nature of the development of the theory, different authors have adopted different terminology. We use the term extending for what Harada [16] and his school refer to as the dual to "lifting module." His terminology is used in [7].
(2) By Zorn's Lemma, every submodule N of M has a closure.

Seemingly independent of the development of the theory of continuous and extending modules outlined above, Goldie considered *complements* in his study of quotient rings [12], [13]. For N a submodule of M, a *complement* of N (in M) is a submodule L of M that is maximal among those submodules H with the property $H \cap N = 0$. A submodule L is a *complement* (in M) if there exists a submodule N of M such that L is a complement of N. Clearly the concepts of "closed" and "complement" are equivalent (see [7, p.6]). Thus "extending" is equivalent to the term "CS" for "complements are summands", used by Chatters and Hajarnavis; they study left CS-rings, i.e., rings R for which R as a left R-module is extending [6].

An extending module M is *continuous* if it has the additional property that every submodule N isomorphic to a direct summand of M is actually a direct summand of M. A module is *quasi-continuous* if it is extending and for every pair M_1, M_2 of submodules of M with $M_1 \cap M_2 = (0)$, $M_1 \oplus M_2$ is a direct summand of M. Thus, as remarked in Mohamed and Muller [23],
injective $\Rightarrow$ continuous $\Rightarrow$ quasi-continuous $\Rightarrow$ extending. Extending modules need not be quasi-continuous [23, Example 2.9]; quasi-continuous modules are not necessarily continuous [23, Proposition 2.2].

More recently the theories of extending modules and other related concepts have developed as interesting generalizations of the concept of injective module and have come to play an important role in the theories of rings and modules. Although this generalization of injectivity is extremely useful, it does not satisfy some desirable properties. For example, a direct summand of an extending module is extending but a direct sum of two or more extending modules is not necessarily extending. Much work has been done to find necessary and sufficient conditions to ensure that the extending property is preserved under direct sums [7], [8], [9], [10]. Relative injectivity, continuous and quasi-continuous modules have been studied in detail (c.f. [2], [17], [19], [26]) and have been related to torsion theory in [4]; the extending property with respect to a torsion theory has had less attention. In earlier work we studied extending, continuous and quasi-continuous modules relative to module classes and in particular studied extending modules relative to torsion and torsion-free module classes, e.g. [9]. Other articles such as [5], [25], [28], [33], concern different aspects of extending modules. The monograph [8] describes the development of the theory further.

In this paper we define the concept of τ-*extending module* relative to a torsion theory τ and consider desirable properties for τ-*extending* modules. In particular, "When is the direct sum of τ-extending modules τ-extending?" If the torsion theory τ is hereditary then it is easy to see that a direct summand of a τ-extending module is τ-extending.

The outline of this article is as follows: The second section contains the definition of τ-extending and some general background material on torsion theories. In the third section we prove some basic results and compare the τ-extending property to other properties. We provide matrix examples that illustrate our results. The fourth section contains decomposition theorems for τ-extending modules; also we demonstrate similarities to and differences from extending modules. In section five, we show that if τ contains the Goldie torsion theory, then τ-extending modules coincide with the extending modules. If τ is a hereditary torsion theory such that "τ-extending" equals "extending", then τ contains the Goldie torsion theory.

We use basic facts from module theory; otherwise the proofs in this article are essentially self-contained.

2 Background on Torsion theories

By a *class* $\mathcal{C}$ of R-modules we mean a collection of R-modules that contains the zero module and that is closed under isomorphism. By a $\mathcal{C}$-*module* we mean a member of $\mathcal{C}$. If $\mathcal{C}$ is a class of R-modules and M is an R-module then a $\mathcal{C}$-*submodule* of M is a submodule N of M such that N belongs to $\mathcal{C}$.

Let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory. Then τ is uniquely determined by its associated torsion class $\mathcal{T}$ of τ-torsion modules. Here the class $\mathcal{T} := \{M \in$

$Mod\text{-}R \mid \tau(M) = M\}$. The class $\mathcal{F}$ refers to the τ-torsion-free class; thus $\mathcal{F} := \{M \in Mod\text{-}R \mid \tau(M) = 0\}$. A module in $\mathcal{T}$ (respectively in $\mathcal{F}$) is called *τ-torsion* (respectively *τ-torsion-free*). The submodule $\tau(M)$ is the unique maximal $\mathcal{T}$-submodule of M, and is called the *τ-torsion submodule* of M. It satisfies $\tau(M/\tau(M)) = 0$, i.e. $M/\tau(M)$ is an $\mathcal{F}$-module and is τ-torsion-free. A module M is *τ-torsion* if $\tau(M) = M$ and is *τ-torsion-free* if $\tau(M) = 0$.

We now introduce definitions analogous to "essential" submodule, "closed" submodule and "extending" module in Definitions 1.1. The new definitions correspond to a given torsion theory τ on $Mod\text{-}R$.

Definitions 2.1. Let τ be a torsion theory and M an R-module. A submodule N of M is *τ-essential* (in M), denoted $N \leq_{e_\tau} M$, if N is essential in M and M/N is τ-torsion, that is, $\tau(M/N) = M/N$. (Originally defined by Tsai, 1965 [11, p.90].)

A submodule N of M is called *τ-closed* (in M) if N has no proper τ-essential extension in M; this property is denoted $N \leq_{c_\tau} M$.

A module M is *τ-extending* if every τ-closed submodule of M is a direct summand of M.

Remark: If N is τ-closed in M, where N is essential in a submodule K of M and K/N is τ-torsion, then $N = K$. Direct summands of a module M are τ-closed in M for every torsion theory τ.

For every torsion theory τ, both the torsion class $\mathcal{T}$ and the torsion-free class $\mathcal{F}$ of R-modules contain the zero module and both are closed under isomorphisms; that is, if $N \in \mathcal{T}$ and $N' \cong N$, then $N' \in \mathcal{T}$, and similarly for $\mathcal{F}$. A *$\mathcal{T}$-submodule* (or *$\mathcal{F}$-submodule*) of M is a submodule N of M such that N belongs to $\mathcal{T}$ (or $\mathcal{F}$). The torsion class $\mathcal{T}$ is closed under homomorphic images, arbitrary direct sums, and extensions by short exact sequences (see [3], [11] or [27, p.139 Proposition 2.1 and 2.2]). The torsion-free class $\mathcal{F}$ is closed under submodules, extensions by short exact sequences and direct products. A torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ for which the torsion class $\mathcal{T}$ is closed under submodules is called a *hereditary* torsion theory. If the torsion-free class $\mathcal{F}$ is closed under homomorphic images; then τ is called *cohereditary*. The reader is referred to [1], [3], [7], [11] and [27] for more background.

Most of our results are true for torsion classes that are not closed under submodules; we assume a torsion theory is not necessarily hereditary unless we specify that it is hereditary.

Definitions 2.2. Let M be an R-module. The *singular* submodule $Z(M)$ of M is the set $Z(M) := \{m \in M : mI = 0$, for some essential right ideal I of R $\}$. If $Z(M) = M$ (respectively, $Z(M) = 0$), then M is a *singular* (respectively *nonsingular*) module. The *Goldie torsion theory*, denoted
$\tau_G = (\mathcal{T}_G, \mathcal{F}_G)$, is given by, for M an R-module, $\tau_G(M)$ is the second singular submodule $Z_2(M)$ of M. That is, $\tau_G(M)$ is the largest submodule $Z_2(M)$ of M such that $Z_2(M)/Z(M) = Z(M/Z(M))$ [14].

The Goldie torsion theory is hereditary [27, Proposition 7.3]. Examples of τ_G-extending modules include semisimple modules, uniform modules, and quasi-injective modules. Every free abelian group of finite rank is a τ_G-extending module. It is easy to see that every extending module is τ_G-extending and every semisimple module is τ-extending.

3 Basic results concerning τ-extending and related conditions

The following Lemma gives some immediate consequences of the definitions that relate the τ-terms to the original terms without τ.

Lemma 3.1. *Let τ be a torsion theory on Mod-R and let N and K be submodules of a module M. Then*
(1) If N is τ-essential in K and K is τ-essential in M, then N is τ-essential in M.
(2) If N is τ-essential in M, then N is essential in M.
(3) If N is closed in M, then N is τ-closed in M.
(4) If M is a τ-extending module, then M is an extending module.
(5) If M is τ-extending, then N is closed in M if and only if N is τ-closed in M.
(6) If N is a direct summand of M, then N is τ-closed in M.

Proof For part (1), since $N \leq_e K \leq_e M$, we have $N \leq_e M$. On the other hand, the τ-torsion modules K/N and M/K form an exact sequence $0 \to K/N \to M/N \to M/K \to 0$. Since every torsion class is closed under extensions by short exact sequences, M/N is τ-torsion. Thus $N \leq_{e_\tau} M$.
Part (2) follows from Definitions 2.1.
For part (3), we suppose that N is a closed submodule of M. If N is not τ-closed in M, then there exists a proper τ-essential extension K of N in M, and so K is an essential extension of N. This is a contradiction to N being closed. Therefore N is τ-closed in M.
Parts (4) and (6) follow immediately from part (3) and the definitions.
For part (5), let M be a τ-extending module. Every closed submodule is τ-closed from part (3). Conversely, let N be a τ-closed submodule in M. Since M is τ-extending, N is a direct summand of M and hence N is closed in M. □

The converses of Lemma 3.1 parts (2), (3), (4) and (6) are not true in general, as the following example shows. We use an extending module M from [23, Example 2.9].

Example 3.2. *Let F be a field, let R be the upper triangular matrix ring over F, let I be the idempotent right ideal of R, let M denote the right R-module R_R and let K be the submodule of M shown below.*

$$M := R_R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}, I := \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}, K := \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}.$$

Define $\mathcal{T}_I := \{N \in Mod\text{-}R : NI = 0\}$. Then $\tau_I := (\mathcal{T}_I, \mathcal{F}_I)$ is a hereditary torsion theory and M is an extending module that is not a τ_I-extending.

Proof Clearly submodules of τ_I-torsion modules are τ_I-torsion, and so τ_I is hereditary. For convenience we show that M is extending. The right submodules of M are the following:

$$M := \{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in F\}, I := \{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in F\},$$

$$K := \{\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} : a, b \in F\}, N_{(x,y)} := \{\begin{bmatrix} 0 & xc \\ 0 & yc \end{bmatrix} : c \in F\},$$

for fixed $(x, y) \in F \times F$. That is, for every pair $x, y \in F$, $N_{(x,y)}$ is a right submodule of M. (Note $N_{(0,0)} = 0$.) We notice that the only submodules of M that are closed are (0), I (as a submodule) and $N_{(x,y)}$ where $y \neq 0$. For $y \neq 0$, $N_{(x,y)} \cap N_{(1,0)} = 0$, and so $N_{(x,y)}$ is not essential in M or in K; hence $N_{(x,y)}$ is closed if $y \neq 0$. Now $K \cap I = N_{(1,0)} \neq 0$; thus K is essential in M and so K is not closed. For $x \neq 0$, $y = 0$, $N_{(x,0)}$ is an essential submodule of I; thus $N_{(x,0)}$ is not closed in M. Also $M = I \oplus N_{(x,y)}$ if $y \neq 0$. Therefore M is extending.

Since $K \leq_e M$ and M/K is not τ_I-torsion, K is not τ_I-essential in M. Since K has no proper τ_I-essential extension in M, K is τ_I-closed in M. However K is not a direct summand of M; hence M is not a τ_I-extending module. □

A further example, this time with a non-hereditary torsion theory, is provided to show that the converses of Lemma 3.1 parts (2), (3), (4) and (6) are not true in general.

Example 3.3. *Let F, R, M, I, K and $N_{(x,y)}$ be as in Example 3.2. Let L denote the idempotent right ideal shown below.*

$$L := N_{(0,1)} = \{\begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} : c \in F\}.$$

Define $\mathcal{T}_L := \{N \in Mod\text{-}R : NL = N\}$. Then $\tau_L := (\mathcal{T}_L, \mathcal{F}_L)$ is not a hereditary torsion theory. M is an extending module that is not τ_L-extending.

Proof As in Example 3.2, M is extending. Here $M = I \oplus N_{(x,y)}$ for $y \neq 0$ where I and $N_{(x,y)}$ are from Example 3.2. On the other hand, since $N_{(1,0)}$ is essential in I, $N_{(1,0)}$ is not closed in M. But $\tau_L(I/N_{(1,0)}) = 0$, and so $I/N_{(1,0)}$ is not τ_L-torsion. Thus $N_{(1,0)}$ is τ_L-closed in M, but is not a direct summand of M. Thus M is not τ_L-extending. Also, K is essential in M; thus K is not closed in M. Since M/K is τ_L-torsion-free, K is τ_L-closed in M, but K is not a direct summand of M. □

Proposition 3.4. *Let τ be a torsion theory, M a module and K a submodule of M such that either M/K is a τ-torsion-free module or K is closed in M. Then K is a τ-closed submodule of M.*

Proof Let K be a submodule of M such that M/K is τ-torsion-free. To show K is τ-closed, suppose that $K \leq_{e_\tau} L$, for some submodule L of M. Then L/K is τ-torsion. On the other hand, $L/K \leq M/K$ implies L/K is τ-torsion-free. Therefore $L = K$ and so K is τ-closed in M. If K is a closed submodule of M, then K is τ-closed by Lemma 3.1 part (3). □

Example 3.5. *The converse of Proposition* 3.4 *is not true in general because the module $N_{(1,0)}$, described in the proof of Example* 3.2 *is τ_I-closed in M, but $N_{(1,0)}$ is not closed in M, and $M/N_{(1,0)}$ is not a τ_I-torsion-free module.*

Lemma 3.6. *Let τ be a hereditary torsion theory. Then a submodule K of M is τ-closed in M if and only if there exists a submodule H of M such that $K \subseteq H \subseteq M$, K is closed in H and $\tau(M/K) = H/K$.*

Proof Assume first that the submodule K of M is τ-closed in M. Let H be a submodule of M such that $\tau(M/K) = H/K$. We show that K is closed in H. Assume that $K \leq_e K' \leq H$ for a submodule K' of H. Then $K'/K \leq H/K$ and, since τ is hereditary, K'/K is a τ-torsion module. Thus K is τ-essential in K', and so $K = K'$, that is, K is closed in H.

Conversely, let $K \subseteq H \subseteq M$, where K is a closed submodule in H and $\tau(M/K) = H/K$. Assume that K is τ-essential in some submodule K' of M, that is, $K \leq_e K'$ and K'/K is a τ-torsion module. Now $K \leq K' \cap H \leq H$ and $K \leq_e K' \cap H$. But K is closed in H, and so $K = K' \cap H$. Also, since K'/K is a τ-torsion module and $\tau(M/K) = H/K$, we have $K'/K \leq H/K$. Thus $K' \leq H$, and so we have $K = K' \cap H = K'$. Therefore K is τ-closed in M. □

Lemma 3.7. *Let τ be a hereditary torsion theory and let M be an R-module. Then every submodule N of M is τ-essential in some τ-closed submodule K of M.*

Proof Let N be a submodule of M. Let $\tau(M/N) = H/N$. Then M/H is τ-torsion-free. Let K be a closed submodule of H such that $N \leq_e K$. Thus $N \leq_e K \leq_c H$. Since $K/N \leq H/N$ and H/N is a τ-torsion module, K/N is τ-torsion. Thus N is τ-essential in K. Now suppose that K is τ-essential in a submodule V of M; that is, V/K is a τ-torsion module and $K \leq_e V$. Then $K \leq_e H \cap V$. Since K is a closed submodule of H, we have $K = H \cap V$. Now $(H+V)/H \cong V/(H \cap V) = V/K$ and $(H+V)/H \leq M/H$ (τ-torsion-free) and so V/K is τ-torsion-free. On the other hand, V/K is a τ-torsion module, and so $K = V$. Therefore K is τ-closed in M. □

Corollary 3.8. *Let τ be a hereditary torsion theory and let M be an R-module. Then M is τ-extending if and only if every submodule of M is τ-essential in a direct summand of M.*

Proof Suppose that M is a τ-extending module. By Lemma 3.7, every submodule N of M is τ-essential in some τ-closed submodule K of M. By hypothesis, K is a direct summand of M.

Conversely, suppose that every submodule of M is τ-essential in a direct summand of M. Let N be a τ-closed submodule of M. By hypothesis, there exists a direct summand A of M such that N is τ-essential in A. Then $N = A$, and so N is a direct summand of M. Thus M is τ-extending. □

Definition 3.9. For N a submodule of M and K a τ-closed submodule of M that contains N as an τ-essential submodule, the submodule K is called a *τ-closure* of N in M, in analogy with Definition 1.1.

Remark: For τ a hereditary torsion theory, Lemma 3.7 implies that every submodule N of M has a τ-closure in M.

Lemma 3.10. *Let τ be a hereditary torsion theory and $M = M_1 \oplus M_2$ for submodules M_1 and M_2 of a module M. Then:*
(1) If A is a τ-closed submodule of M_1, then A is τ-closed in M.
(2) If M is τ-extending, then M_1 and M_2 are both τ-extending.

Proof For part (1), by Lemma 3.6, A is closed in some submodule B_1 of M_1 such that $\tau(M_1/A) = B_1/A$. Then M_1/B_1 is a τ-torsion-free module. Let $B = B_1 \oplus B_2$, where $B_2 = \tau(M_2)$. Hence A is a closed submodule of B and $M/B = M_1/B_1 \oplus M_2/B_2$, a direct sum of τ-torsion-free modules, and so M/B is τ-torsion-free. Now $A \subseteq B \subseteq M$, A is closed in B and $\tau(M/A) = B/A$. Therefore, by Lemma 3.6, A is τ-closed in M.
Part (2) follows from part (1). □

Proposition 3.11. *Let τ be a hereditary torsion theory and let M be a τ-torsion module. Then M is extending if and only if M is τ-extending.*

Proof For the "only if" direction, assume that M is an extending module and let N be a τ-closed submodule of M. Let K be a closure of N in M. Then $N \leq_e K \leq_c M$. By hypothesis, K is a direct summand of M. Since M/N is τ-torsion, K/N is τ-torsion. Thus N is τ-essential in K, and so $N = K$ and N is a direct summand of M. Thus M is τ-extending. The converse is clear by Lemma 3.1, part (4). □

Theorem 3.12. *Let τ be a hereditary torsion theory. The following statements are equivalent for an R-module M.*
(1) M is τ-extending.
(2) Every submodule is τ-essential in a direct summand of M.
(3) M is extending and if K is a submodule of M such that M/K is τ-torsion-free, then K is a direct summand of M.

Proof The equivalence of items (1) and (2) is Corollary 3.8. For (1) ⇒ (3), M is extending by Lemma 3.1 part (4).

For the other part of (3), let K be a submodule of M such that M/K is τ-torsion-free. By (2), there exists a direct summand T of M such that K is τ-essential in T. Then T/K is both a τ-torsion and a τ-torsion-free module. Thus $K = T$, and so K is a direct summand of M.

For (3) ⇒ (1), let N be a τ-closed submodule of M. By Lemma 3.6, there exists a submodule H of M such that $N \subseteq H \subseteq M$, N is closed in H and $\tau(M/N) = H/N$. Thus $\tau(M/H) = 0$ and so by (3), H is a direct summand of M. Hence, H is extending and therefore N is a direct summand of H and so of M. Thus M is τ-extending. □

Theorem 3.13. *Let τ be a torsion theory and M be an R-module. Assume that every τ-torsion-free module is projective. Then:*
(1) $M = \tau(M) \oplus N$, for some submodule N of M.
(2) If M is an extending module, then M is τ-extending.

Proof For part (1), since $M/\tau(M)$ is a τ-torsion-free module, it is projective. Thus $\tau(M)$ is a direct summand of M.

For part (2), let M be an extending module and let K be a τ-closed submodule of M. By Zorn's Lemma, let T be a closure of K in M, so that $K \leq_e T \leq_c M$. Thus T is a direct summand of M. If $K = T$, then K is a direct summand of M and we are done. Assume that $K \neq T$. Since K is a τ-closed submodule of M, T/K is a τ-torsion-free module. Then by hypothesis, T/K is projective. Thus K is a direct summand of T, and so K is a direct summand of M. □

4 Decompositions of τ-extending modules

Let U and M both be R-modules. We say that U is *M-injective* if, given a submodule N of M, every homomorphism $\varphi : N \to U$ can be lifted to a homomorphism $\theta : M \to U$ such that $\theta(x) = \varphi(x)$, for all $x \in N$. A class of R-modules $\{M_i \mid i \in I\}$, where I is an index set, is called *relatively injective* if M_i is M_j-injective, for every pair of distinct $i, j \in I$.

We use the following Lemma from Kamal and Muller:

Lemma 4.1. *[22, Lemma 17] Let a module $M = M_1 \oplus M_2$ be a direct sum of submodules M_1, M_2. If M_2 is M_1-injective, then for each submodule N of M with $N \cap M_2 = 0$, there exists a submodule M' of M such that $M = M' \oplus M_2$ and $N \subseteq M'$.*

This decomposition for a τ-extending module is analogous to [18, Theorem 8].

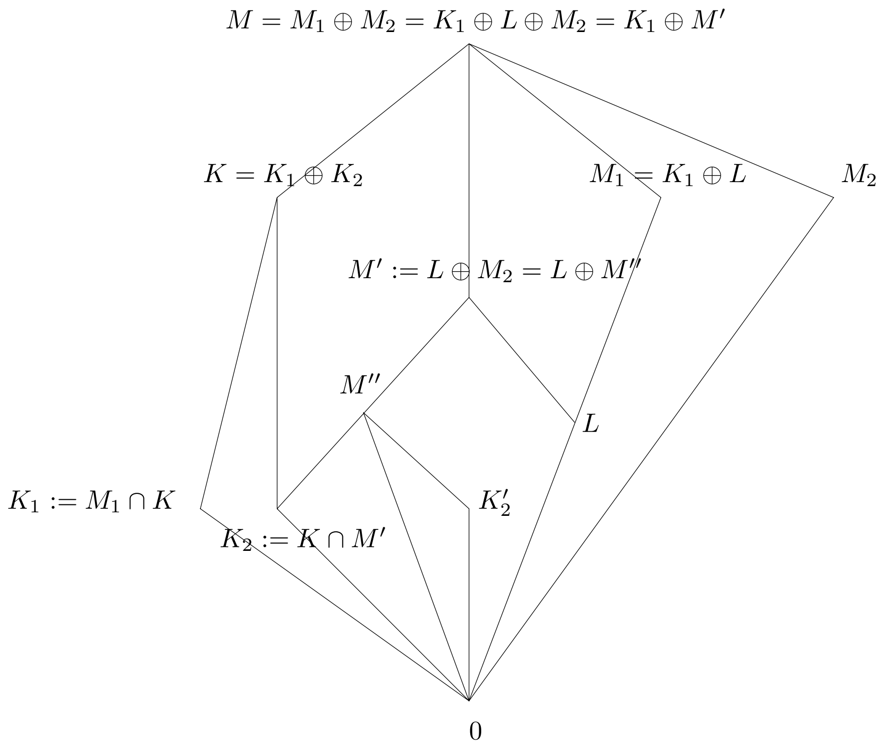
Theorem 4.2. *Let τ be a hereditary torsion theory and let M be an R-module that is a direct sum $M = M_1 \oplus M_2$ of two relatively injective submodules M_1 and M_2. Then M is τ-extending if and only if both M_1 and M_2 are τ-extending.*

Proof Necessity is clear by Lemma 3.10, part (2).

Conversely, suppose that M_1 and M_2 are both τ-extending modules. By Lemma 3.1, part (4), M_1 and M_2 are extending and by [18, Theorem 8], M is extending. Now let K be a submodule of M such that M/K is τ-torsion-free. Let $K_1 := M_1 \cap K$. Note that M_1/K_1 is τ-torsion-free, since $M_1/K_1 \cong M_1/(M_1 \cap K) \cong (M_1 + K)/K \subseteq M/K$.

By hypothesis and Theorem 3.12, K_1 is a direct summand of M_1. Hence $M_1 = K_1 \oplus L$ for some submodule L of M_1. Set $M' := L \oplus M_2$ and $K_2 := K \cap M'$. See diagram below. Then we have
$K = K \cap M = K \cap (M_1 \oplus M_2) = K \cap (K_1 \oplus L \oplus M_2) = K \cap (K_1 \oplus M')$.
Now $K_1 \subseteq K$ and thus by modularity $K = K_1 \oplus (K \cap M') = K_1 \oplus K_2$.
Moreover, K_2 is a submodule of M' such that $K_2 \cap L = 0$, because
$K_2 \cap L \subseteq K \cap (L \oplus M_2) \cap L \subseteq K \cap (L \oplus M_2) \cap M_1 \subseteq (K \cap M_1) \cap (L \oplus M_2) = K_1 \cap (L \oplus M_2) = 0$, since $(K_1 \oplus L) \oplus M_2 = M_1 \oplus M_2$ is a direct sum.

Since M_1 is M_2-injective, we have L is M_2-injective. Thus, by Lemma 4.1, there exists a submodule $M'' \leq M'$ such that $M' = L \oplus M''$ and $K_2 \leq M''$. Note that $M'' \cong M_2$ since also $M' = L \oplus M_2$.



Claim: M''/K_2 is τ-torsion-free. To see this, note $M/K = (K_1 \oplus L \oplus M'')/(K_1 \oplus K_2)$, and so we can write $M/K = ((K_1 \oplus L)/K_1) \oplus (M''/K_2) \cong L \oplus (M''/K_2)$. Now M/K is τ-torsion-free, and therefore M''/K_2 is τ-torsion-free.

Since the class of τ-extending modules is closed under isomorphism and $M_2 \cong M''$, we have that M'' is τ-extending. By Theorem 3.12, it follows that K_2 is a direct summand of M''. Thus $M'' = K_2 \oplus K_2'$ for some submodule K_2' of M''. Now $M' = L \oplus M'' = L \oplus K_2 \oplus K_2'$ and so $M = K_1 \oplus M' = K_1 \oplus L \oplus K_2 \oplus K_2' = K \oplus L \oplus K_2'$. Hence K is a direct summand of M. Now by Theorem 3.12, M is τ-extending. □

We also give a decomposition of τ-extending modules as an analogue of [21, Theorem 1].

Theorem 4.3. *Let τ be a hereditary torsion theory and M an R-module. If M is τ-extending then $M = \tau(M) \oplus N$, where $\tau(M)$ and N are relatively injective τ-extending modules.*

Proof Suppose that M is a τ-extending module.

Claim 1: $\tau(M) \leq_{c_\tau} M$. Proof of Claim 1: Suppose there exists a submodule N_1 of M such that $\tau(M)$ is τ-essential in N_1. Then $N_1/\tau(M)$ is τ-torsion. Since $M/\tau(M)$ is τ-torsion-free, $N_1/\tau(M)$ is τ-torsion-free. This implies $N_1 = \tau(M)$. Thus $\tau(M)$ is τ-closed in M.

By hypothesis, $\tau(M)$ is a direct summand of M. Thus, for some submodule N of M, we have $M = \tau(M) \oplus N$. By Lemma 3.10 (2), $\tau(M)$ and N are both τ-extending modules.

Claim 2: $\tau(M)$ is an N-injective module. Proof of Claim 2: For a submodule K of N, let $f : K \to \tau(M)$ be a homomorphism. We want to extend f to $N \to \tau(M)$. Define $K'=\{k - f(k) : k \in K\}$. By Lemma 3.7, there exists a τ-closed submodule X of M such that $K' \leq_{e_\tau} X \leq_{c_\tau} M$. By the definition of τ-extending, $M = X \oplus X'$ for some submodule X' of M. We see that $K' \cap \tau(M) = 0$. Therefore $X \cap \tau(M) = 0$ because K' is essential in X. Thus K' and X are τ-torsion-free, that is, $\tau(X) = 0$. Now $\tau(M) = \tau(X) + \tau(X') = \tau(X') \leq X'$. Let $Y' := N \cap X'$. By modularity, $X' = \tau(M) \oplus Y'$, since $\tau(M) \subseteq X'$. For the canonical epimorphism $\eta : M = X \oplus \tau(M) \oplus Y' \to \tau(M)$, define $g := \eta\mid_N$. For $k \in K \subseteq N$, $g(k) = \eta\mid_N (k)$. Now $K' \leq X$ and so for every $k \in K$, $k - f(k) \in X$ and $\eta(X) = 0$; thus $\eta(k - f(k)) = 0$. Thus $\eta(k) = \eta(f(k)) = f(k)$ because $f(k) \in \tau(M)$ and $\eta\mid_{\tau(M)}= 1_{\tau(M)}$. Thus f extends to $g : N \to \tau(M)$. Therefore $\tau(M)$ is an N-injective module.

It is obvious that N is $\tau(M)$-injective because N is τ-torsion-free and $\tau(M)$ is τ-torsion, and so every homomorphism from a submodule of $\tau(M)$ into N is the 0 homomorphism, which obviously extends to $\tau(M)$. □

Theorem 4.4. *Let τ be a torsion theory and let M be an R-module such that $M = \tau(M) \oplus N$, where N is a semisimple submodule of M. If $\tau(M)$ and N are relatively injective τ-extending modules, then M is τ-extending.*

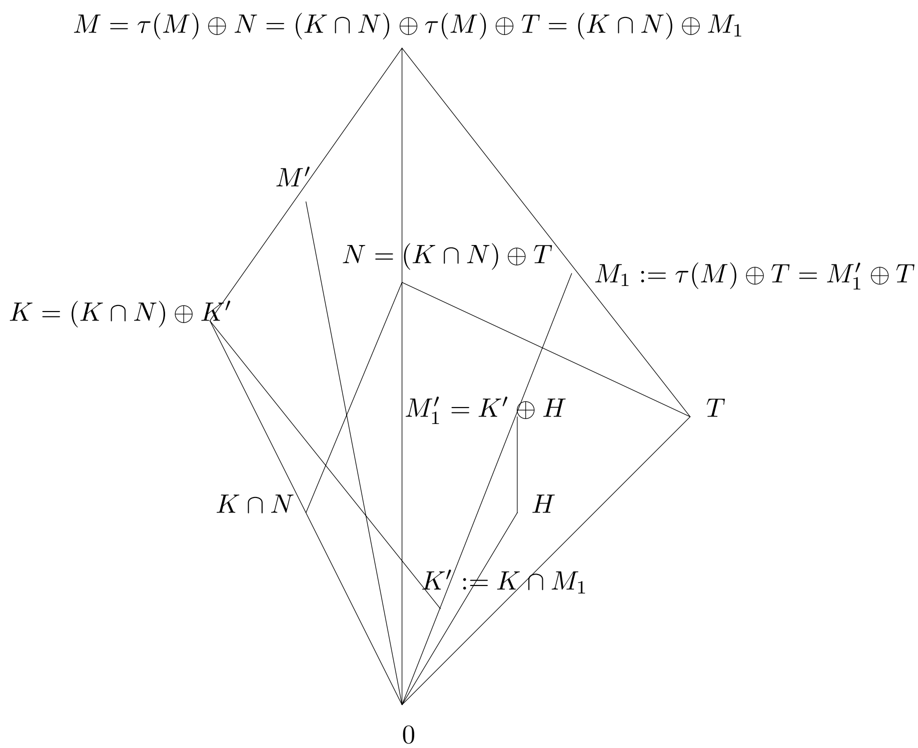
Proof Let K be a τ-closed submodule of M. We show K is a direct summand of M.

Case 1: If $K \cap \tau(M) = 0$, then we use Lemma 4.1, since $\tau(M)$ is N-injective; there exists a submodule U of M such that $M = U \oplus \tau(M)$ and $K \subseteq U$. Since U is semisimple, K is a direct summand of U and so K is a direct summand of M and we are done.

Case 2: Suppose $K \cap N = 0$. Since N is $\tau(M)$-injective, again by Lemma 4.1 there exists a submodule M' of M such that $M = M' \oplus N$ and $K \subseteq M'$. Since $M = \tau(M) \oplus N$, we have $M' \cong \tau(M)$, and so M' is τ-extending. Since K is τ-closed in M, K is τ-closed in M'. Hence K is a direct summand of M' and so K is a direct summand of M, and we are done in this case.

Case 3: Assume $K \cap N \neq 0$. Then $N = (K \cap N) \oplus T$, for some submodule T of N, since N is semisimple. Since $M = \tau(M) \oplus (K \cap N) \oplus T$, by modularity we see that $K = (K \cap N) \oplus [K \cap (\tau(M) \oplus T)]$.

Let $M_1 := \tau(M) \oplus T$ and $K' := K \cap M_1$. (See diagram.) Since K' is a direct summand of K, Lemma 3.1 part (6) implies that K' is a τ-closed submodule of K.



Claim: K' is τ-closed in M_1. Proof: Suppose not. Then there exists an L with $K' \leq_e L \leq M_1$ and $\tau(L/K') = L/K'$. Then $K = (K \cap N) \oplus K' \leq_e (K \cap N) \oplus L \leq (K \cap N) \oplus M_1 = M$ since $M = (K \cap N) \oplus \tau(M) \oplus T$. Now $((K \cap N) \oplus L)/((K \cap N) \oplus K') \cong L/K'$, a contradiction since K is τ-closed. Therefore K' is τ-closed in M_1 and the claim holds. (Thus $L = K'$.) Also $K' \cap T = 0$, because $K' \cap T \subseteq K \cap T \subseteq K \cap N \cap T = 0$.

Since T is a direct summand of N and N is $\tau(M)$-injective, by [1, Proposition 16.10] T is $\tau(M)$-injective. By Lemma 4.1, there exists a submodule M_1' of M_1 such that $M_1 = M_1' \oplus T$ and $K' \leq M_1'$. Then $M_1' \cong \tau(M)$ and so M_1' is τ-extending. Since K' is τ-closed in M_1, K' is τ-closed in M_1' and so K' is a direct summand of M_1'. Then $M_1' = K' \oplus H$ for some submodule H of M_1'. We write $M = (K \cap N) \oplus \tau(M) \oplus T = (K \cap N) \oplus M_1 = (K \cap N) \oplus M_1' \oplus T = (K \cap N) \oplus K' \oplus H \oplus T = K \oplus H \oplus T$. Thus M is τ-extending. □

Corollary 4.5. *Let τ be a torsion theory and let M be an R-module such that $M = M_1 \oplus M_2$ where M_1 is τ-torsion and M_2 is a semisimple submodule. If M_1 is τ-extending and M_1 and M_2 are relatively injective submodules, then M is τ-extending.*

Proof Since M_2 is semisimple, $M_2 = \tau(M_2) \oplus N$, where N is a τ-torsion-free module and $M_1 = \tau(M_1)$. This implies $M = \tau(M_1) \oplus \tau(M_2) \oplus N = \tau(M) \oplus N$. Thus M is τ-extending by Theorem 4.4. □

5 Goldie related conditions

In this section we compare τ-extending modules relative to two different torsion theories. By Lemma 3.1 part (4), every τ-extending module is extending, but Example 3.2 shows the converse is not true in general. We show in Theorem 5.3 that if the torsion theory τ contains the Goldie torsion theory τ_G, defined in Definition 2.2, then every extending module is τ-extending.

Lemma 5.1. *Let τ_G be the Goldie torsion theory. Then every extending module is τ_G-extending.*

Proof Let M be an extending module and let N be a τ_G-closed submodule of M. We claim that N is a closed submodule of M. If not, assume that there is an essential extension K of N in M, i.e. $N \leq_e K \leq M$. Then K/N is singular. Thus N is τ_G-essential in K, a contradiction. Thus N is a closed submodule of M. By hypothesis, N is a direct summand of M. □

For torsion theories ρ and τ, we write $\rho \leq \tau$ provided every ρ-torsion R-module is a τ-torsion R-module.

Proposition 5.2. *Let τ and ρ be torsion theories such that $\rho \leq \tau$. If an R-module M is ρ-extending, then M is τ-extending.*

Proof Let M be a ρ-extending R-module. Let K be a τ-closed submodule of M. Claim: K is ρ-closed in M. If there exists a submodule T of M such that T is a ρ-essential extension of K, then T/K is a ρ-torsion module and $K \leq_e T$. Hence T/K is a τ-torsion module and so $T = K$, since K is τ-closed in M. Thus K is ρ-closed in M, and the claim is proved. By hypothesis K is a direct summand of M. Thus M is τ-extending. □

Theorem 5.3. *Let τ be a torsion theory with $\tau_G \leq \tau$. Then the following conditions are equivalent for an R-module M.*
(1) M is τ-extending.
(2) M is extending.
(3) M is τ_G-extending.

Proof (1) ⇒ (2): This follows from Lemma 3.1 part (4).
(2) ⇒ (3): This follows from Lemma 5.1.
(3) ⇒ (1): This follows from Proposition 5.2. □

Question 5.4. Let τ be a torsion theory. If every extending module is τ-extending, does τ contain the Goldie torsion theory?

If the torsion theory is hereditary, the answer is "yes".

Theorem 5.5. *Let τ be a hereditary torsion theory on Mod-R. Then every extending module is τ-extending if and only if $\tau_G \leq \tau$.*

Proof Sufficiency is clear by Theorem 5.3. Conversely, suppose that every extending module is τ-extending. First we show that every singular module is τ-torsion. So let M be a nonzero singular R-module. There exists an R-module F and an essential submodule K of F such that M is isomorphic to F/K by [15, Proposition 3.26]). Let E denote the injective hull of F. Suppose that E/K is not a τ-torsion module. Then there exists a proper submodule L of E containing K such that $\tau(E/K) = L/K$ and thus $(E/K)/(L/K) \cong E/L$ is τ-torsion-free. Since injective modules are extending, using the hypothesis, we see E is a τ-extending module. Thus by Theorem 3.12 (3), L is a direct summand of E. This contradicts the fact that L is essential in E. Thus E/K is a τ-torsion module. Since the torsion theory τ is hereditary and $M \cong F/K \leq E/K$, we have F/K is τ-torsion. Thus M is τ-torsion. Now let M be a nonzero R-module with $\tau_G(M) = M$. Then $M/Z(M)$ is singular. Since $Z(M)$ is also singular, $M/Z(M)$ and $Z(M)$ are τ-torsion from what we have proved. Thus M is τ-torsion. It follows that $\tau_G \leq \tau$. □

We let ξ denote the trivial torsion theory, where the torsion class of modules consists of only the zero module. The improper torsion theory χ is such that the torsion class is all of Mod-R, i.e., χ=(Mod-R,0).

Proposition 5.6. *Let ξ denote the trivial torsion theory and let χ denote the improper torsion theory. Let M be an R-module. Then:*
(1) M is ξ-extending if and only if M is semisimple.
(2) M is extending if and only if M is χ-extending.
(3) M is τ_G-extending if and only if M is χ-extending.

Proof Part (1) and (2) are clear from the definitions.
For part (3), since $\tau_G \leq \chi$, Proposition 5.2 shows τ_G-extending implies χ-extending. If M is χ-extending then M is extending by part (2), and so M is τ_G-extending by Lemma 5.1. □

Proposition 5.7. *Let τ be a cohereditary torsion theory, that is, $\mathcal{F}$ is closed under homomorphic images. If an R-module M is τ-torsion-free and τ-extending, then M is semisimple.*

Proof Let N be a submodule of M. We claim that N is τ-closed in M. Since M is τ-torsion-free and τ is cohereditary, M/N is τ-torsion-free. By Proposition 3.4, N is τ-closed in M. Thus N is a direct summand of M. Therefore M is a semisimple module. □

While reading an early draft of this article, P.F. Smith asked the following question, since extending modules generalize the concept of injective for modules.

Question 5.8. Let R be a ring. Does there exist a torsion theory τ such that the class of injective R-modules coincides with the class of τ-extending modules?

We don't know of a torsion theory that answers this question yet but the following theorem may help.

Definition 5.9. An R-module M is said to be τ-*injective* if, for each short exact sequence $0 \longrightarrow L \longrightarrow X \longrightarrow N \longrightarrow 0$ of R-modules L, X, N, where N is τ-torsion, the sequence $Hom_R(X, M) \longrightarrow Hom_R(L, M) \longrightarrow 0$ is exact (see [4]).

Theorem 5.10. *Let τ be a hereditary torsion theory and let τ_G be the Goldie torsion theory on Mod-R. The following conditions are equivalent.*
(1) $\tau_G \leq \tau$.
(2) Every extending module is τ-extending.
(3) Every τ-injective module is injective.

Proof (1) $\Leftrightarrow$ (2) by Theorem 5.5.
(1) $\Leftrightarrow$ (3) by [4, Theorem 2.1]. □

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