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GEOMETRY OF THE RANGE OF A VECTOR MEASURE

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Abstract

A.A. Lyapunov [18] in 1940 proved that the range of a countably additive bounded measure with values in a finite dimensional vector space is compact and, in the non-atomic case, is convex. Simplified proofs, unified versions, topological versions, generalizations and related theory have appeared in literature from time to time and recently in a series of papers by D. E. Wulbert, for instance, [21]. [16] gives a comprehensive critical survey. In this paper the range of a two dimensional vector measure with emphasis on geometry, particularly, its boundary is studied.

1 Introduction

For the sake of simplicity of presentation we confine our attention to probability measures most of the time and leave the adaptation to general measure to the reader.

In Section 2 we give some basic definitions and a few useful facts about convex functions.

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Section 3 is devoted to the study of the range of a two dimensional vector measure. Emphasis is on its boundary. For instance, let $\boldsymbol{\mu} = (\mu_1, \mu_2)$ be a pair of probability measures on a measurable space (X, Ω), then the *lower and upper bound of the range of* $\boldsymbol{\mu}$ *are* $F^1_{\boldsymbol{\mu}} : \mu_1(\Omega) \to \mathbb{R}$ and $F^2_{\boldsymbol{\mu}} : \mu_1(\Omega) \to \mathbb{R}$ given by $F^1_{\boldsymbol{\mu}}(x) = \inf\{\mu_2 E : E \in \Omega, \mu_1 E = x\}$ and $F^2_{\boldsymbol{\mu}}(x) = \sup\{\mu_2 E : E \in \Omega, \mu_1 E = x\}$ for each $x \in \mu_1(\Omega)$. Clearly $F^2_{\boldsymbol{\mu}}(x) = 1 - F^1_{\boldsymbol{\mu}}(1-x)$. We prove the following results:

(a) If μ_1 is non-atomic, then $F^2_{\boldsymbol{\mu}}$ and $F^1_{\boldsymbol{\mu}}$ are increasing.

(b) If μ_1 and μ_2 are both non-atomic, then $F^1_{\boldsymbol{\mu}}$ is an absolutely continuous convex function on $[0, 1]$.

Let Σ be a σ-algebra of subsets of $[0, 1]$ with $\mathcal{B} \subset \Sigma \subset \mathcal{M}$, $\mathcal{B}$ is the class of Borel subsets of $[0, 1]$ and $\mathcal{M}$, the class of Lebesegue measurable subsets of $[0, 1]$.

In Section 4 we define a finite measure μ on $([0, 1], \Sigma)$ with $\Sigma = \mathcal{B}$ or $\mathcal{M}$ to be *right expanding* if

$$\mu(E + a) \geq \mu E \quad \text{for all} \quad E \in \Sigma, a \geq 0 \quad \text{with} \quad E + a \subset [0, 1]\,.$$

Let m denote the Lebesgue measure and μ be a non-atomic probability measure on $([0, 1], \mathcal{B})$. We show that there exists a unique Borel measure λ on $([0, 1], \mathcal{B})$ which is right expanding and is absolutely continuous with respect to m such that

$$\lambda[0, x] = F^1_{(m,\mu)}(x) \quad \text{for all} \quad x \in [0, 1]\,.$$

Further in case μ is right expanding and absolutely continuous with respect to m, then we have $\lambda = \mu$.

Similar results can be proved for certain vector measures on some general measure space (X, Ω).

In Section 5 we give different examples to illustrate our results and to show the significance of conditions in the hypotheses in our results.

2 Basic results on convex functions and the cumulative distribution function of a Borel measure

We begin with the definition and a few facts on convex functions based on fundamental results from sources such as [3], [7], [14], [17], [19] and [20].

Let $\mathbb{N}$ be the set of positive integers, $\mathbb{R}$ be the set of real numbers with the usual topology and $\mathbf{I}$ an interval in $\mathbb{R}$. Let $\mathbf{I}_0$ denote the interior of $\mathbf{I}$. Let $\mathbb{R}_e = \mathbb{R} \cup \{\infty, -\infty\}$ be the extended real number system. We equip $\mathbb{R}_e$ with the topology arising from the metric $d(x, y) = |\tan^{-1} x - \tan^{-1} y|$ where $\tan^{-1} \infty$ and $\tan^{-1}(-\infty)$ are taken as $\frac{\pi}{2}$ and $-\frac{\pi}{2}$, respectively.

2.1 Definition. A function $f : \mathbf{I} \to \mathbb{R} \cup \{\infty\}$ is said to be *convex* if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \text{ for all } x, y \in \mathbf{I} \text{ and } 0 \leq \lambda \leq 1\,.$$

A function $f : \mathbf{I} \to \mathbb{R} \cup \{-\infty\}$ is said to be *concave* if the function $(-f)$ is convex.

2.2 Remark. (i) Let $f : \mathbf{I} \to \mathbb{R}$ be a convex function. Then we have the following:

(a) If $\alpha, \beta \in \mathbf{I}$ are such that $f(\alpha) \leq f(x)$ for x in $[\alpha, \beta]$, then putting $\gamma = \sup\{x \in [\alpha, \beta] : f(\alpha) = f(x)\}$, we have that $f(x) = f(\alpha)$ for x in $[\alpha, \gamma]$ and f is strictly increasing on $[\gamma, \beta]$. As a consequence f is increasing on $[\alpha, \beta]$.

(b) The function f is absolutely continuous on each closed interval $[a, b]$ contained in the interior $\mathbf{I}_0$ of $\mathbf{I}$.

(c) The right- and left-derivatives of f exist at each point of $\mathbf{I}_0$ and are equal except on a countable set.

(d) The right- and left-derivative are monotone increasing functions and at each point of $\mathbf{I}_0$ the left-derivative is less than or equal to the right-derivative.

(ii) Let $\mathbf{A}$ be a convex subset of $\mathbb{R}^2$ and let D be the projection of $\mathbf{A}$ on the x-axis. Suppose that $\inf\{y : (x, y) \in A\} > -\infty$ for each x in D, then the function $f : D \to \mathbb{R}$ defined by $f(x) = \inf\{y : (x, y) \in \mathbf{A}\}$ is a convex function.

2.3 Remark. Let $-\infty < a < b < \infty$.

(i) If $f : [a, b) \to \mathbb{R}$ is convex and increasing, then f is continuous on $[a, b)$.

(ii) If $f : (a, b] \to \mathbb{R}$ is convex and decreasing, then f is continuous on $(a, b]$.

2.4 Definition. Let $f : \mathbf{I} \to \mathbb{R}$ be a function and x a point of $\mathbf{I}$, f is said to be *convex, concave, monotone increasing, monotone decreasing or absolutely continuous near* x if there exists $\alpha > 0$ such that f is convex, concave, monotone increasing, monotone decreasing or absolutely continuous respectively on $(x - \alpha, x + \alpha) \cap \mathbf{I}$.

2.5 Remark. Let $f : [a, b] \to \mathbb{R}$ be a convex function.

(i) If f is absolutely continuous near 'a' and near 'b', then f is absolutely continuous on $[a, b]$.

(ii) If f is monotone increasing near 'a' and monotone decreasing near 'b', then f is absolutely continuous on $[a, b]$.

(iii) If f is continuous and monotone near 'a' and 'b', then f is absolutely continuous on $[a, b]$.

2.6 Let $f : \mathbf{I} \to \mathbb{R}$ and $x \in \mathbf{I}_0$, we define the *second symmetric* or *second Schwarz derivative* $f^{[11]}(x)$ of f at x by

$$f^{[11]}(x) = \lim_{h \to 0} \frac{(f(x+h) - 2f(x) + f(x-h))}{h^2}.$$

Even if this limit does not exist, we can define the lower and upper second symmetric (or Schwarz) derivatives by taking the limit inferior and limit superior and denote them by $\underline{f}^{[11]}(x)$ and $\bar{f}^{[11]}(x)$, respectively.

Clearly if f is differentiable in some neighbourhood of the point $x \in \mathbf{I}_0$ and f'' exists at 'x', then $f''(x) = f^{[11]}(x)$.

2.7 Remark. Let $f : \mathbf{I} \to \mathbb{R}$ be a continuous function.

(i) If f is convex on $\mathbf{I}_0$, then it is so on $\mathbf{I}$.

(ii) If $\underline{f}^{[11]}(x) \geq 0$ on I_0 then f is convex.

2.8 Let μ be a finite measure on $([0,1], \Sigma)$. We shall denote by F_μ, the cumulative distribution function of μ, that is $F_\mu(x) = \mu[0,x]$ for $x \in [0,1]$

2.9 Remark. (i) F_μ is a monotone increasing function which is continuous on the right. Moreover if $\mu\{a\} = 0$ for all singleton set $\{a\}$ in Σ, then F_μ is continuous on $[0,1]$.

(ii) The measure μ is absolutely continuous with respect to the Lebesgue measure if and only if F_μ is absolutely continuous.

2.10 Borel equivalence Let X be a complete separable metric space and $\mathcal{B}_X$ be the Borel σ-algebra on X. Let μ_1 be a complete non-atomic probability measure defined on a σ-algebra $\mathcal{T}_{\mu_1}$ containing $\mathcal{B}_X$ and μ_2 be a non-atomic probability measure on $(X, \mathcal{T}_{\mu_1})$. Then there exists a mapping

$$\varphi : X \to [0,1]$$

such that under this mapping the measure space $(X, \mathcal{T}_{\mu_1}, \mu_1)$ is isomorphic to $([0,1], \mathcal{M}, m)$. That is

$$\varphi(B) \in \mathcal{M} \text{ for all } B \in \mathcal{T}_{\mu_1},\ \varphi^{-1}(E) \in \mathcal{T}_{\mu_1} \text{ for all } E \in \mathcal{M},$$
$$\mu_1(B) = m(\varphi(B)) \text{ for all } B \in \mathcal{T}_{\mu_1}$$

and

$$\mu_1(\varphi^{-1}(E)) = mE \text{ for all } E \in \mathcal{M}.$$

Put $\psi(E) = \mu_2(\varphi^{-1}(E))$ for all $E \in \mathcal{M}$. Then ψ is a non-atomic probability measure on $([0,1], \mathcal{M})$ and if μ_2 is absolutely continuous with respect to μ_1, then ψ is absolutely continuous with respect to m.

3 Geometry of the range of a two-dimensional vector measure

3.1 By a *measure pair* $\boldsymbol{\mu}$ on a measurable space (X, Ω), we mean

$$\boldsymbol{\mu} = (\mu_1, \mu_2),$$

where μ_1 and μ_2 are signed measures on the measurable space (X, Ω).

If μ_1 and μ_2 are both finite then $\boldsymbol{\mu}$ will be called a two dimensional vector measure.

3.2 Definition. Let $\boldsymbol{\mu} = (\mu_1, \mu_2)$ be a measure pair on (X, Ω). Then we define the *lower* and *upper bounds* of the range of μ viz.,

$$F^1_{\boldsymbol{\mu}} : \mu_1(\Omega) \to \mathbb{R}_e \quad \text{and} \quad F^2_{\boldsymbol{\mu}} : \mu_1(\Omega) \to \mathbb{R}_e$$

by $F^1_{\boldsymbol{\mu}}(x) = \inf\{\mu_2 E : E \in \Omega, \mu_1 E = x\}$ and $F^2_{\boldsymbol{\mu}}(x) = \sup\{\mu_2 E : E \in \Omega, \mu_1 E = x\}$ for $x \in \mu_1(\Omega)$.

In the case when $\mu_1 = m$ and $\mu_2 = \mu$, we will simply write F^1_μ and F^2_μ in place of $F^1_{\boldsymbol{\mu}}$ and $F^2_{\boldsymbol{\mu}}$.

Let $\rho(x) = -x$, $x \in \mathbb{R}_e$. Clearly $F^2_{-\boldsymbol{\mu}} = -F^1_{\boldsymbol{\mu}} \circ \rho$ and $F^1_{-\boldsymbol{\mu}} = -F^2_{\boldsymbol{\mu}} \circ \rho$.

The following Remark puts these concepts in the right perspective with discussion and examples in [6], [4] and [9] as the essential background.

3.3 Remark. Let $\boldsymbol{\mu} = (\mu_1, \mu_2)$ be a measure pair on (X, Ω).

(i) Let $\mathbb{R}(\boldsymbol{\mu})$ be the range of $\boldsymbol{\mu}$, that is $\mathbb{R}(\boldsymbol{\mu}) = \{(\mu_1 E, \mu_2 E) : E \in \Omega\}$ treated as a subset of $\mathbb{R}_e \times \mathbb{R}_e$.

- (a) For $(x, y) \in \mathbb{R}(\boldsymbol{\mu})$ we have $F^1_{\boldsymbol{\mu}}(x) \le y \le F^2_{\boldsymbol{\mu}}(x)$.
- (b) For any $x \in \mu_1(\Omega)$, if $F^1_{\boldsymbol{\mu}}(x) > -\infty$ then $(x, F^1_{\boldsymbol{\mu}}(x)) \in$ Bd. $\mathbb{R}(\boldsymbol{\mu})$ and if $F^2_{\boldsymbol{\mu}}(x) < \infty$ then $(x, F^2_{\boldsymbol{\mu}}(x)) \in$ Bd. $\mathbb{R}(\boldsymbol{\mu})$.
- (c) In case both μ_1 and μ_2 are non-atomic finite measures, then by Liapounov's theorem $\mathbb{R}(\boldsymbol{\mu})$ is compact and convex, as a consequence (b) is applicable and the graph of the lower and upper bounds of the range of $\boldsymbol{\mu}$ are actually the lower and upper boundary of the range of $\boldsymbol{\mu}$.

(ii) (a) If μ_1, μ_2 are both non-negative finite measures then

$$F^2_{\boldsymbol{\mu}}(x) = \mu_2(X) - F^1_{\boldsymbol{\mu}}(\mu_1(X) - x) \quad \text{for} \quad x \in \mu_1(\Omega).$$

(b) If μ_1, μ_2 are probability measures, then $\mathbb{R}(\boldsymbol{\mu})$ is symmetric about $\left(\frac{1}{2}, \frac{1}{2}\right)$ and

$$F^2_{\boldsymbol{\mu}}(x) = 1 - F^1_{\boldsymbol{\mu}}(1 - x) \quad \text{for} \quad x \in \mu_1(\Omega).$$

(iii) Let $C(\boldsymbol{\mu})$ be the range of the function $E \to (\mu_1 E, \mu_2(X \sim E)), E \in \Omega$, the notation reminiscent of the division of a cake into pieces E and $X \sim E$ for pre-assigned values viewed by two persons P_1 and P_2 measured by μ_1 and μ_2 respectively. We define the functions $C^1_{\boldsymbol{\mu}}$ and $C^2_{\boldsymbol{\mu}}$ on $\mu_1(\Omega)$ by

$$C^1_{\boldsymbol{\mu}}(x) = \inf\{\mu_2(X \sim E) : E \in \Omega, \mu_1 E = x\};$$

$$C^2_{\boldsymbol{\mu}}(x) = \sup\{\mu_2(X \sim E) : E \in \Omega, \mu_1 E = x\} \quad \text{for each} \quad x \in \mu_1(\Omega).$$

(a) If $\boldsymbol{\mu}$ is finite then we have

$$C(\boldsymbol{\mu}) = \{(x, -y) : (x, y) \in \mathbb{R}(\boldsymbol{\mu})\} + (0, \mu_2(X)),$$

$$C^1_{\boldsymbol{\mu}}(x) = \mu_2(X) - F^2_{\boldsymbol{\mu}}(x) \quad \text{and} \quad C^2_{\boldsymbol{\mu}}(x) = \mu_2(X) - F^1_{\boldsymbol{\mu}}(x)\,.$$

Thus it is enough to confine our attention to $\mathbb{R}(\boldsymbol{\mu})$ and $F^1_{\boldsymbol{\mu}}$ in this case.

(b) Further, if μ_1 and μ_2 are both probability measures then $E \in \Omega$ for P_1 gives a good cut if $\mu_1 E \geq \frac{1}{2}$ and $\mu_2(X - E) \geq \frac{1}{2}$. Thus our interest is in the set

$$G(\boldsymbol{\mu}) = C(\boldsymbol{\mu}) \cap \left\{(x, y) : x \geq \frac{1}{2}, y \geq \frac{1}{2}\right\}$$

and some optimal solutions. Equivalently, we can look at its reflection in the line $y = \frac{1}{2}$, that is the set

$$G_r(\boldsymbol{\mu}) = \mathbb{R}(\boldsymbol{\mu}) \cap \left\{(x, y) : x \geq \frac{1}{2}, y \leq \frac{1}{2}\right\}.$$

This, in turn, leads us to study $F^1_{\boldsymbol{\mu}}$ and $F^2_{\boldsymbol{\mu}}$ restricted to $\left[\frac{1}{2}, 1\right]$, or, in view of (ii) (b) above to these functions restricted to $\left[0, \frac{1}{2}\right]$ only.

Our next Remark is an adaptation of the standard measure theory arguments used for reducing some general cases to non-atomic cases ([5], [9], [11], [12] and [13]).

3.4 Remark. Let $\boldsymbol{\mu} = (\mu_1, \mu_2)$ be a measure pair on (X, Ω) such that μ_2 possesses the Lebesgue decomposition (μ_0, μ_3) with respect to μ_1, that is $\mu_2 = \mu_0 + \mu_3$ where $\mu_0 \perp \mu_1$ and $\mu_3 << \mu_1$. Let $\boldsymbol{\mu}_a = (\mu_1, \mu_3)$. Then, we have

(i) $\mathbb{R}(\boldsymbol{\mu}) = \mathbb{R}(\boldsymbol{\mu}_a) + \{0\} \times \mathbb{R}(\mu_0)$,

(ii) $F^1_{\boldsymbol{\mu}} = F^1_{\boldsymbol{\mu}_a} + \inf\{\mu_0 E : E \in \Omega\}$ and $F^2_{\boldsymbol{\mu}} = F^2_{\boldsymbol{\mu}_a} + \sup\{\mu_0 E : E \in \Omega\}$. In particular, if μ_0 is non-negative, then

$$F^1_{\boldsymbol{\mu}} = F^1_{\boldsymbol{\mu}_a} \quad \text{and} \quad F^2_{\boldsymbol{\mu}} = F^2_{\boldsymbol{\mu}_a} + \mu_0 X,$$

(iii) If $\mathbb{R}(\boldsymbol{\mu}_a)$ is a convex subset of $\mathbb{R}^2$ and $\mu_0(\Omega)$ is an interval in $\mathbb{R}$, then $\mathbb{R}(\boldsymbol{\mu})$ is a convex subset of $\mathbb{R}^2$.

3.5 Proposition. *Let $\boldsymbol{\mu} = (\mu_1, \mu_2)$ be a measure pair of non-negative measures on (X, Ω). Suppose μ_1 is finite and non-atomic and put $\mu_1(\Omega) = [0, \mu_1 X] = \mathbf{I}$. Then $F^1_{\boldsymbol{\mu}}$ and $F^2_{\boldsymbol{\mu}}$ are both increasing on $\mathbf{I}$.*

Proof Let $0 \leq x_1 < x_2 \leq \mu_1 X$. As μ_1 is a finite non-atomic measure there exist sets E_1, E_2 with $\mu_1 E_1 = x_1$ and $\mu_1 E_2 = x_2$. Further, for any such pair (E_1, E_2), there exists a pair (G_1, G_2) of sets with $E_1 \subset G_2$ and $G_1 \subset E_2$ satisfying $\mu_1 G_1 = x_1$ and $\mu_1 G_2 = x_2$. So $F^1_{\boldsymbol{\mu}}(x_1) \leq F^1_{\boldsymbol{\mu}}(x_2)$ and $F^2_{\boldsymbol{\mu}}(x_1) \leq F^2_{\boldsymbol{\mu}}(x_2)$. □

3.6 Proposition. *Let $\boldsymbol{\mu} = (\mu_1, \mu_2)$ be a two-dimensional vector measure on (X, Ω).*

(i) *If $\mathbb{R}(\boldsymbol{\mu})$ is convex, then $F^1_{\boldsymbol{\mu}}$ is a convex function on $\mu_1(\Omega)$ and $F^2_{\boldsymbol{\mu}}$ is a concave function on $\mu_1(\Omega)$.*

(ii) *If $\mathbb{R}(\boldsymbol{\mu})$ is convex and μ_1 and μ_2 are both non-negative then $F^1_{\boldsymbol{\mu}}$ and $F^2_{\boldsymbol{\mu}}$ are both increasing functions on $[0, \mu_1 X]$.*

(iii) *In particular, if both μ_1 and μ_2 are non atomic probability measures, then $F^1_{\boldsymbol{\mu}}$ is an increasing convex function and $F^2_{\boldsymbol{\mu}}$ is an increasing concave function.*

Proof

(i) The results follow from Remark 2.2.

(ii) We first use (i) and apply Remark 2.2 (i)(a) to $f = F^1_{\boldsymbol{\mu}}$ with $\alpha = 0 = f(\alpha)$ to obtain that $F^1_{\boldsymbol{\mu}}$ is increasing. The rest is now immediate from Remark 3.3.

(iii) We appeal to Liapounov's Theorem and apply (i) and (ii) above. □

3.7 Proposition. *Let $\boldsymbol{\mu} = (\mu_1, \mu_2)$ be a two dimensional vector measure on (X, Ω) with μ_1 and μ_2 both non-negative.*

(i) *If $\mathbb{R}(\boldsymbol{\mu})$ is convex and closed, then $F^1_{\boldsymbol{\mu}}(x)$ is absolutely continuous on $[0, \mu_1 X]$.*

(ii) *In particular, it is so if μ_1 and μ_2 are both non-atomic probability measures.*

Proof (i) It is enough to give the proof when μ_1 is a probability measure. Proposition 3.6 gives that $F^1_{\boldsymbol{\mu}}(x)$ is increasing and convex on $[0, 1]$. Thus by Remark 2.3 $F^1_{\boldsymbol{\mu}}$ is continuous on $[0, 1)$. Now we shall show that $F^1_{\boldsymbol{\mu}}$ is also continuous at 1. Since $\mathbb{R}(\boldsymbol{\mu})$ is closed and convex in $\mathbb{R}^2$, by Remark 3.3

we have for each $n \in \mathbb{N}$, $\left(1-\frac{1}{n}, F^1_{\boldsymbol{\mu}}\left(1-\frac{1}{n}\right)\right) \in \mathbb{R}(\boldsymbol{\mu})$. So $\left(1, F^1_{\boldsymbol{\mu}}(1-)\right) = \lim \left(1-\dfrac{1}{n}, F^1_{\boldsymbol{\mu}}\left(1-\dfrac{1}{n}\right)\right) \in \mathbb{R}(\boldsymbol{\mu})$. Therefore, there exists $E \in \Omega$ such that $\mu_1 E = 1$ and $\mu_2 E = F^1_{\boldsymbol{\mu}}(1-)$. Hence $F^1_{\boldsymbol{\mu}}(1-) \geq F^1_{\boldsymbol{\mu}}(1)$. But $F^1_{\boldsymbol{\mu}}$ is an increasing function, therefore $F^1_{\boldsymbol{\mu}}(1) \geq F^1_{\boldsymbol{\mu}}(1-)$. Thus $F^1_{\boldsymbol{\mu}}(1) = F^1_{\boldsymbol{\mu}}(1-)$. Therefore $F^1_{\boldsymbol{\mu}}$ is continuous at 1 and hence by Remark 2.5 it is absolutely continuous on $[0,1]$.

(ii) It follows from Liapounov's Theorem and part (i). □

3.8 Remark. Various generalization of a convex function can be found in literature. For instance, S. J. Dilworth, R. Howard and J. W. Roberts [8] have defined an approximately convex function to be a function f on an interval $\mathbf{I}$ to $\mathbb{R}$ satisfying

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} + 1 \quad \text{for all} \quad x, y \in \mathbf{I}\,.$$

They have carried out an extensive study of such functions. Clearly for any probability measures μ_1 and μ_2 on (X, Ω) with $\mu_1(\Omega) = [0,1], F^1_{\boldsymbol{\mu}}$ and $F^2_{\boldsymbol{\mu}}$ are both approximately convex functions on $[0,1]$. We can use their work to our advantage in the case when μ_1 and μ_2 are both finite measures and $\mu_1(\Omega)$ is an interval.

3.9 Remark. For a subset A of a normed linear space X, there are various notions to measure the degree of non-convexity of A. We recall two of them defined by V. M. Kadets [15] and J. Elton and T. P. Hill [10] who use these numbers for the range of a vector measure μ to measure the degree of non atomicity of μ in their own ways. One may look at [1] and [2] for $\| \,.\, \|_p$ - norms and sharper bounds.

(a) [15] defines $C(\mathbf{A}) = \sup\left\{d\left(\frac{1}{2}(x+y), \mathbf{A}\right) : x, y \in \mathbf{A}\right\}$, which is zero if and only if $\bar{\mathbf{A}}$ is convex.

(b) [10] defines $D(\mathbf{A}) = \sup\{d(x, \mathbf{A}) : x \in \mathrm{Co}(\mathbf{A})\}$, which in case $\mathbf{A}$ is closed, is zero if and only if $\mathbf{A}$ is convex. Here $Co(\mathbf{A})$ denotes the convex hull of $\mathbf{A}$.

(c) Clearly $C(\mathbf{A}) \leq D(\mathbf{A})$. If $\mathbf{A}$ consists of the vertices of an equilateral triangle of unit side, then $C(\mathbf{A}) = \frac{1}{2}$ and $D(\mathbf{A}) = \frac{1}{\sqrt{3}}$.

For the sake of convenience we take empty sums to be zero.

4 Right expanding measures

4.1 Definition. Let μ be a finite measure on $([0,1], \Sigma)$ with $\Sigma = \mathcal{B}$ or $\mathcal{M}$, then μ is said to be right expanding if

$$\mu(E+a) \geq \mu E \quad \text{for all} \quad E \in \Sigma,\ a \geq 0 \ \text{ with } \ E + a \subset [0,1].$$

4.2 Proposition. *If a finite measure μ on $([0,1],\Sigma)$ is right expanding, then all the four Dini's derivatives D^+F_μ, D_+F_μ, D^-F_μ and D_-F_μ on their respective domains are monotone increasing.*

Proof It follows immediately from the definition. □

We have a partial converse of the above Proposition.

4.3 Proposition. *Let μ be a finite measure on $([0,1],\Sigma)$ with $\Sigma = \mathcal{B}$ or $\mathcal{M}$. If μ is absolutely continuous with respect to m and D^+F_μ is monotone increasing, then μ is right expanding.*

Proof By Remark 2.9(i) F_μ is an increasing real-valued function, therefore F_μ is differentiable a.e. and $F'_\mu = D^+F_\mu$ a.e.. By Remark 2.9(ii) F_μ is absolutely continuous, therefore it is the definite integral of its derivative. So for $0 \le x \le 1$, we have

$$\mu[0,x] = F_\mu(x) = \int_0^x D^+F_\mu dm.$$

Let (α,β), an open sub-interval of $[0,1]$ and $a > 0$ be such that $(\alpha,\beta)+a \subset [0,1]$. Then as $\mu\{\beta\} = \mu\{\beta+a\} = 0$ and D^+F_μ is increasing, we have

$$\begin{aligned}\mu(\alpha,\beta) &= \mu(\alpha,\beta] = \int_\alpha^\beta D^+F_\mu dm \le \int_{\alpha+a}^{\beta+a} D^+F_\mu dm \\ &= F_\mu(\beta+a) - F_\mu(\alpha+a) = \mu(\alpha+a,\beta+a] \\ &= \mu(\alpha+a,\beta+a) = \mu((\alpha,\beta)+a).\end{aligned}$$

Thus

$$\mu(\alpha,\beta) \le \mu((\alpha,\beta)+a). \tag{1}$$

Let $E \in \Sigma$ and $a > 0$ be such that $E + a \subset [0,1]$. We shall show that $\mu E \le \mu(E+a)$. Let $\varepsilon > 0$ be given. Since $\mu \ll m$, there exists $\delta > 0$ such that for any $A \in \Sigma$ with $mA < \delta$, we have $\mu A < \varepsilon$. Also there exist disjoint open intervals of $[0,1]$, say $I_1, I_2, \ldots, I_n$ such that

$$U = \bigcup_{i=1}^n I_i, \qquad m(E\Delta U) < \delta.$$

Therefore, we have

$$\mu E \le \mu(E\cup U) = \mu(E\cap U) + \mu(E\Delta U) \le \mu U + \varepsilon. \tag{2}$$

Since m is translation invariant, we have

$$m((E\sim U)+a) + m((U\sim E)+a) < \delta.$$

So

$$\begin{aligned}\mu(E+a) &\geq \mu((E\cap U)+a) = \mu(U+a) - \mu((U\sim E)+a)\\ &\geq \mu(U+a)-\varepsilon \end{aligned} \tag{3}$$

And we have, in view of (1),

$$\mu(U+a) = \sum_{i=1}^{n} \mu(I_i + a) \geq \sum_{i=1}^{n} \mu I_i = \mu U.$$

Using this in (3), we get

$$\mu(E+a) \geq \mu U - \varepsilon. \tag{4}$$

From (4) and (2), we get $\mu E \leq \mu(E+a)+2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $\mu E \leq \mu(E+a)$. Hence μ is right expanding. □

4.4 Proposition. *Let μ be a finite measure on $([0,1],\Sigma)$ which is right expanding and absolutely continuous with respect to m, then F^1_μ is the cumulative distribution function of μ, that is*

$$F_\mu(x) = F^1_\mu(x) \ \ \textit{for all} \ \ x\in[0,1]\,.$$

Proof For each $x \in [0,1]$, we have

$$F^1_\mu(x) \ = \ \inf\{\mu E : E\in\Sigma, mE = x\} \leq \mu[0,x] = F_\mu(x).$$

To prove the reverse inequality, we shall show that

$$F_\mu(x) = \mu[0,x] \leq \mu E \ \ \text{for all} \ \ E\in\Sigma \ \ \text{with} \ \ mE = x\,.$$

Let (α,β) be any open sub-interval of $[0,1]$, then since μ is right expanding we have $\mu(0,\beta-\alpha) \leq \mu(\alpha,\beta)$. Also $\mu \ll m$. So $\mu\{x\} = 0$ for any $x\in[0,1]$.

Now let $\{(a_i,b_i)\}_{i=1}^n$ be a finite family of disjoint open sub-intervals of $[0,1]$. Without loss of generality we may assume that

$$0 \leq a_1 \leq b_1 \leq a_2 < b_2 \leq \ldots \leq a_i < b_i \leq a_{i+1} < \ldots \leq a_n < b_n \leq 1.$$

We have

$$\begin{aligned}&\mu\left(0, \sum_{i=1}^{n}(b_i - a_i)\right)\\ &\quad = \ \mu\Bigg[(0, b_1 - a_1)\cup((b_1-a_1),(b_1-a_1)+(b_2-a_2))\cup\ldots\\ &\qquad\qquad \cup\Bigg(\sum_{i=1}^{n-1}(b_i-a_i), \sum_{i=1}^{n}(b_i-a_i)\Bigg)\Bigg]\end{aligned}$$

$$= \quad \mu(0, b_1 - a_1) + \sum_{k=2}^{n} \left[\mu(\sum_{i=1}^{k-1}(b_i - a_i), \sum_{i=1}^{k}(b_i - a_i)) \right] \tag{5}$$

Since $\sum_{i=2}^{k}(a_i - b_{i-1}) \geq 0$ for each $k = 2, 3, \ldots, n$, we have, for any such k,

$$\begin{aligned} &\mu \left(\sum_{i=1}^{k-1}(b_i - a_i), \sum_{i=1}^{k}(b_i - a_i) \right) \\ &\leq \quad \mu \left(\sum_{i=1}^{k-1}(b_i - a_i) + \sum_{i=2}^{k}(a_i - b_{i-1}), \sum_{i=1}^{k}(b_i - a_i) + \sum_{i=2}^{k}(a_i - b_{i-1}) \right) \\ &= \quad \mu(a_k - a_1, b_k - a_1) \leq \mu(a_k, b_k)\,. \end{aligned}$$

Therefore, from (5) we get

$$\mu \left(0, \sum_{i=1}^{n}(b_i - a_i) \right) \leq \mu(a_1, b_1) + \sum_{k=2}^{k} \mu(a_k, b_k) = \sum_{i=1}^{n} \mu(a_i, b_i)$$

Thus

$$\mu \left(\bigcup_{i=1}^{n}(a_i, b_i) \right) \geq \mu \left(0, m \left(\bigcup_{i=1}^{n}(a_i, b_i) \right) \right). \tag{6}$$

Now let $E \in \Sigma$ and $\varepsilon > 0$ be given. Since $\mu \ll m$, there exists $\delta > 0$ such that for every $A \in \Sigma$ with $mA \leq \delta$ we have $\mu A < \varepsilon$. Also for the above $\delta > 0$, there exist disjoint open intervals $I_1, I_2, I_3, \ldots, I_n$ in $[0, 1]$ such that if $U = \bigcup_{i=1}^{n} I_i$, then $m(E \Delta U) < \delta$.

Using (6), we have

$$\begin{aligned} \mu E \geq \mu(E \cap U) \quad &= \quad \mu(U \sim (U \sim E)) = \mu U - \mu(U \sim E) \\ &> \quad \mu U - \varepsilon \geq \mu(0, mU) - \varepsilon \end{aligned} \tag{7}$$

Thus, $|mU - mE| \leq m(U \sim E) + m(E \sim U) < \delta$. Therefore, we get

$$mU - \delta < mE < mU + \delta \tag{8}$$

Now $m(mE - \delta, mE) = \delta$, therefore $\mu(mE - \delta, mE) < \varepsilon$. Using (7) and (8), we get $\mu(0, mE) - 2\varepsilon < \mu E$. Since $\varepsilon > 0$ is arbitrary and $\mu\{0\} = 0 = \mu\{mE\}$, we have $\mu[0, mE] \leq \mu E$. Therefore, for all $x \in [0, 1]$, $F_\mu(x) \leq F_\mu^1(x)$. Hence $F_\mu(x) = F_\mu^1(x)$ for all $x \in [0, 1]$.

4.5 Theorem. *Let μ be a non-negative measure on $([0, 1], \mathcal{B})$.*

(i) *Suppose $\mathbb{R}(m, \mu)$ is a closed convex subset of $\mathbb{R}^2$. Then the following hold.*

(a) *There exists a unique Borel measure λ on $([0, 1], \mathcal{B})$ which is right expanding and is absolutely continuous with respect to m such that*

$F_\lambda = F^1_\mu$.

(b) *In case μ is absolutely continuous with respect to m, we have $\lambda = \mu$ if and only if μ is right expanding if and only if $F_\mu = F^1_\mu$.*

(ii) *Suppose μ possesses the Lebesgue decomposition $\mu = \mu_0 + \mu_a$ with $\mu_0 \perp m$, $\mu_a \ll m$ and $\mathbb{R}(m, \mu_a)$ is a closed convex subset of R^2, then (i) (a) holds and $\lambda = \mu_a$ if and only if μ_a is right expanding if and only if $F_{\mu_a} = F^1_\mu$.*

In particular, the conclusions in (i) *hold if μ is a non-atomic probability measure. The conclusions in* (ii) *hold if μ_a is a non-atomic probability measure.*

Proof (i) (a) By Proposition 3.6 and Proposition 3.7, F^1_μ is an increasing, convex and absolutely continuous function on $[0,1]$ with $F^1_\mu(0) = 0$. Therefore there exists a unique Borel measure λ such that for all 'a' and 'b' in $[0,1]$ with $a < b$, we have

$$\lambda(a, b] = F^1_\mu(b) - F^1_\mu(a) \quad \text{and} \quad \lambda\{0\} = 0\,.$$

So $F_\lambda = F^1_\mu$ which is absolutely continuous on $[0,1]$. Therefore, by Remark 2.9, λ is absolutely continuous with respect to m. Now as F^1_μ is a convex function, by Remark 2.2, $D^+F^1_\mu = D^+F_\lambda$ is a monotone increasing function. Thus, by Proposition 4.3, λ is right expanding. To establish part (b) we apply Proposition 4.4.

The first part of (ii) follows from Remark 3.4. For the second part we apply Proposition 4.4 to μ_a. □

4.6 Remark. If f is a non-negative continuous function on $[0,1]$, which is increasing with $f(0) \neq f(1)$, then the measure μ on Σ given by

$$\mu E = \int_E f dm$$

is right expanding and the boundary of $\mathbb{R}(m, \mu)$ is not smooth at $(0,0)$ and $(1,1)$. This is because

$$\begin{aligned} F^1_\mu(0) &= F^2_\mu(0) = 0, F^1_\mu(1) = F^2_\mu(1), \\ F^{1\prime}_\mu(0) &= f(0) \neq f(1) = F^{2\prime}_\mu(0) \end{aligned}$$

and $\quad F^{1\prime}_\mu(1) = f(1) \neq f(0) = F^{2\prime}_\mu(1)$.

4.7 Remark. Let X be a complete, separable metric space and $\mathcal{B}_X$ the Borel σ-algebra on X. Let μ_1 be a complete non-atomic probability measure defined on a σ-algebra $\mathcal{T}_{\mu_1}$ containing $\mathcal{B}_X$ and μ_2 be a non-atomic probability measure on $(X, \mathcal{T}_{\mu_1})$ such that $\mu_2 \ll \mu_1$. Put $\boldsymbol{\mu} = (\mu_1, \mu_2)$.

Let $\varphi : X \to [0,1]$ be an isomorphism of $(X, \mathcal{T}_{\mu_1}, \mu_1)$ onto $([0,1], \mathcal{M}, m)$ and let $F : [0,1] \to \mathbb{R}$ be given by $F(x) = \mu_2(\varphi^{-1}[0,x])$ for all $x \in [0,1]$.

If D^+F is increasing then $F^1_{\boldsymbol{\mu}}(x) = F(x)$ for all $x \in [0,1]$.

5 Examples

Now we discuss well known different types of examples to illustrate our results and to show the significance of conditions in the hypotheses in our results.

5.1 Examples of two-dimensional vector measures that are not non-atomic but have a convex and closed range

Let $\{r_n\}_{n=1}^{\infty}$ be a sequence of distinct points in $[0,1]$.

(a) Let $\mu_1 = m$ and $\mu_2 = \frac{1}{2}\left(m + \sum_{n=1}^{\infty} 1/2^n \delta_{r_n}\right)$ be measures on $([0,1],\Sigma)$. Then $\mu_2 = \mu_0 + \mu_3$ where $\mu_0 = \dfrac{1}{2}\displaystyle\sum_{n=1}^{\infty} 1/2^n \delta_{r_n}$ and $\mu_3 = \dfrac{1}{2}m$. Now, we have $\mu_0 \perp \mu_1$ and $\mu_3 \ll \mu_1$. That is, (μ_0, μ_3) is the Lebesgue decomposition of μ_2 with respect to μ_1. Further $\mu_1(\Sigma) = \mu_2(\Sigma) = [0,1]$; $\mu_0(\Sigma) = \mu_3(\Sigma) = [0, \dfrac{1}{2}]$. Let $\boldsymbol{\mu} = (\mu_1, \mu_2)$. Then $\boldsymbol{\mu}_a = (\mu_1, \mu_3)$. Then the range $\mathbb{R}(\boldsymbol{\mu})$ is the parallelogram in $\mathbb{R}^2$ bounded by the lines $x = 0$, $y = \frac{1}{2}x$, $x = 1$ and $y = \frac{1}{2}x + \frac{1}{2}$. We see that the range is convex and compact although the measure $\boldsymbol{\mu}$ has atoms. Further, we have $F^1_{\boldsymbol{\mu}}(x) = \frac{1}{2}x$ and $F^2_{\boldsymbol{\mu}}(x) = F^2_{\boldsymbol{\mu}_a}(x) + \mu_0 X = \frac{1}{2}x + \frac{1}{2}$, $0 \le x \le 1$. We see that $F^1_{\boldsymbol{\mu}}(x)$ and $F^2_{\boldsymbol{\mu}}(x)$ are both absolutely continuous and increasing and $F^1_{\boldsymbol{\mu}}$ is the cumulative distribution function of the right expanding measure $\lambda = \mu_1 = \frac{1}{2}m$. Moreover, the boundary of $\mathbb{R}(\boldsymbol{\mu})$ is smooth except at $(0,0)$, $\left(1, \frac{1}{2}\right)$, $(1,1)$ and $\left(0, \frac{1}{2}\right)$. Finally $\mathbf{G}(\boldsymbol{\mu})$ is a two-dimensional closed convex set expressing bias towards the first person P_1.

(b) We consider the measure pair $\boldsymbol{\upsilon} = (\upsilon_1, \upsilon_2)$, where

$$\upsilon_1 = \mu_2 = \frac{1}{2}\left(m + \sum_{n=1}^{\infty} 1/2^n \delta_{r_n}\right) \text{ and } \upsilon_2 = \mu_1 = m.$$

Then $\upsilon_2 \ll \upsilon_1$. Using part (a) above we get that the range $\mathbb{R}(\boldsymbol{\upsilon})$ is the parallelogram in $\mathbb{R}^2$ bounded by the lines $y = 0$, $y = 2x - 1$, $y = 1$ and $y = 2x$. Here also the range is convex and compact although the measure $\boldsymbol{\upsilon}$ has atoms. We also have

$$F^1_{\boldsymbol{\upsilon}}(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{2} \\ 2x - 1, & \frac{1}{2} \le x \le 1. \end{cases}$$

and

$$F^2_{\boldsymbol{\upsilon}}(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 1, & \frac{1}{2} \le x \le 1. \end{cases}$$

So $F^1_{\boldsymbol{\upsilon}}$ and $F^2_{\boldsymbol{\upsilon}}$ are both absolutely continuous and increasing. Further $F^1_{\boldsymbol{\upsilon}}$ is the cumulative distribution function of the measure η given by $d\eta = 2\chi_{[1/2,1]}dm$ which is right expanding and absolutely continuous with respect to m. Moreover, the boundary of $\mathbb{R}(\boldsymbol{v})$ is smooth except at $(0,0)$, $\left(\frac{1}{2},0\right)$, $(1,1)$ and $\left(\frac{1}{2},1\right)$. Finally $\mathbf{G}(\boldsymbol{v})$ is a two-dimensional closed convex set expressing bias towards the second person P_2.

5.2 Examples of two-dimensional vector measures with disconnected range having two convex components

(a) Let $\mu_1 = m$ and $\mu_2 = \frac{1}{2}(m+\delta_1)$ be measures on $([0,1],\Sigma)$ and $\boldsymbol{\mu} = (\mu_1,\mu_2)$. Then $\mu_2 = \mu_0 + \mu_3$, where $\mu_0 = \frac{1}{2}\delta_1$ and $\mu_3 = \frac{1}{2}m$. Now, we have $\mu_0 \perp \mu_1$ and $\mu_3 \ll \mu_1$. That is (μ_0,μ_3) is the Lebesgue decomposition of μ_2 with respect to μ_1. So $\boldsymbol{\mu}_a = (\mu_1,\mu_3) = (m, \frac{1}{2}m)$. Further,

$$\mu_1(\Sigma) = \mu_2(\Sigma) = [0,1]; \mu_0(\Sigma) = \{0, \frac{1}{2}\} \text{ and } \mu_3(\Sigma) = [0, \frac{1}{2}].$$

So the range of $\boldsymbol{\mu} = (\mu_1,\mu_2)$ consists of the two line segments $y = \frac{1}{2}x$ and $y = \frac{1}{2}(x+1)$, $x \in [0,1]$. And, for $x \in [0,1]$ we have $F^1_{\boldsymbol{\mu}}(x) = \frac{1}{2}x$ and $F^2_{\boldsymbol{\mu}}(x) = \frac{1}{2}(x+1)$. The range $\mathbb{R}(\boldsymbol{\mu})$ of $\boldsymbol{\mu}$ is not convex but has two disjoint compact convex components. However, the range $R(\boldsymbol{\mu}_a)$ consists of the line segment $y = \frac{1}{2}x$, $0 \le x \le 1$ and is thus convex and closed. The upper and lower bounds of the range, viz. $F^1_{\boldsymbol{\mu}}, F^2_{\boldsymbol{\mu}}$, are both monotonically increasing, absolutely continuous and convex on $[0,1]$. Moreover, μ_1, μ_2 and μ_3 are all right expanding and $F^1_{\boldsymbol{\mu}}$ is the cumulative distribution function of the measure $\lambda = \mu_3 = \frac{1}{2}m$. Finally, $\mathbf{G}(\boldsymbol{\mu})$ is a one-dimensional closed convex set expressing bias towards the first person P_1.

(b) Consider the measure pair $\boldsymbol{\nu} = (\nu_1,\nu_2)$, where

$$\nu_1 = \mu_2 = \frac{1}{2}(m+\delta_1) \text{ and } \nu_2 = \mu_1 = m\,.$$

Here $\nu_2 \ll \nu_1$. Using (a) above, the range $\mathbb{R}(\boldsymbol{\nu})$ consists of two line segments $y = 2x$, $0 \le x \le \frac{1}{2}$ and $y = 2x-1$, $\frac{1}{2} \le x \le 1$. Thus the range has two disjoint compact convex components but it is not convex. Therefore, we have

$$F^1_{\boldsymbol{\nu}}(x) = \begin{cases} 2x, & 0 \le x < \frac{1}{2} \\ 2x-1, & \frac{1}{2} \le x \le 1 \end{cases} \text{ and } F^2_{\boldsymbol{\nu}}(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 2x-1, & \frac{1}{2} < x \le 1\,. \end{cases}$$

Thus the upper bound and the lower bound of the range are continuous on $[0,1] \sim \{\frac{1}{2}\}$ and neither of them is increasing and thus cannot be the cumulative distribution function of any measure. Further, $F^1_{\boldsymbol{\nu}}$ is right continuous at $\frac{1}{2}$ and $F^2_{\boldsymbol{\nu}}$ is left continuous at $\frac{1}{2}$, though both have jump discontinuity at $\frac{1}{2}$. Finally, $\mathbf{G}(\boldsymbol{\nu})$ is a one-dimensional closed convex set expressing bias towards the second person P_2. Moreover, if $\mathbf{A} = \mathbb{R}(\boldsymbol{\mu})$ or $\mathbb{R}(\boldsymbol{\nu})$, then $C(\mathbf{A}) = D(\mathbf{A}) = \frac{1}{2\sqrt{5}}$.

5.3 Examples of two-dimensional vector measures with connected range expressible as union of two convex sets

Let $\{r_n\}_{n=1}^{\infty}$ be any sequence of distinct points in the interval $[0,1]$ and $r_1 = 0$.

(a) Let $\mu_1 = \dfrac{1}{2}(m+\delta_0)$, $\mu_2 = \displaystyle\sum_{n=1}^{\infty} 1/2^n \delta_{r_n}$ be measures on $([0,1],\Sigma)$. Then $\mu_2 = \mu_0 + \mu_3$ where $\mu_0 = \displaystyle\sum_{n=2}^{\infty} 1/2^n \delta_{r_n}$ and $\mu_3 = \frac{1}{2}\delta_0$. We next note that $\mu_0 \perp \mu_1$ and $\mu_3 \ll \mu_1$. That is (μ_0, μ_3) is the Lebesgue decomposition of μ_2 with respect to μ_1. So $\boldsymbol{\mu}_a = (\mu_1, \mu_3)$ for $\boldsymbol{\mu} = (\mu_1, \mu_2)$. We have

$$\mu_1(\Sigma) = \mu_2(\Sigma) = [0,1], \mu_0(\Sigma) = [0, \frac{1}{2}] \quad \text{and} \quad \mu_3(\Sigma) = \{0, \frac{1}{2}\}.$$

Now we shall evaluate $\mathbb{R}(\boldsymbol{\mu}), F^1_{\boldsymbol{\mu}}$ and $F^2_{\boldsymbol{\mu}}$. We consider three cases.

(i) $0 \le x < \frac{1}{2}$. If $E \in \Sigma$ is such that $\mu_1 E < \frac{1}{2}$, then $0 \notin E$, and, therefore, $\mu_3 E = 0$ and $\mu_2 E = \displaystyle\sum_{n=2}^{\infty} 1/2^n \delta_{r_n} E \le \frac{1}{2}$. Thus $([0,\frac{1}{2}) \times [0,1]) \cap \mathbb{R}(\boldsymbol{\mu}_a) = [0,\frac{1}{2}) \times \{0\}$. So $[0,\frac{1}{2}) \times [0,1] \cap \mathbb{R}(\boldsymbol{\mu}) = [0,\frac{1}{2}) \times [0,\frac{1}{2}]$, $F^1_{\boldsymbol{\mu}}(x) = 0$ and $F^2_{\boldsymbol{\mu}}(x) = \dfrac{1}{2}$ for $0 \le x < \dfrac{1}{2}$.

(ii) $x = \frac{1}{2}$. Now $\mu_1 E = x$ if and only if either $0 \in E$ and $mE = 0$ or $0 \notin E$ and $mE = 1$. In the former case $\mu_3 E = \frac{1}{2}$ and in the later case $\mu_3 E = 0$. So we have $\{\frac{1}{2}\} \times [0,1] \cap \mathbb{R}(\boldsymbol{\mu}_a) = \{(\frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2})\}$. Thus, $\{\frac{1}{2}\} \times [0,1] \cap \mathbb{R}(\boldsymbol{\mu}) = \{\frac{1}{2}\} \times [0,1]$. Further, $F^1_{\boldsymbol{\mu}}(\frac{1}{2}) = 0$ and $F^2_{\boldsymbol{\mu}}(\frac{1}{2}) = 1$.

(iii) $\frac{1}{2} < x \le 1$ If $E \in \Sigma$ is such that $\mu_1 E > \frac{1}{2}$, then $0 \in E$ and therefore $\mu_3 E = \frac{1}{2}$. Thus $(\frac{1}{2}, 1] \times [0,1] \cap \mathbb{R}(\boldsymbol{\mu}_a) = (\frac{1}{2}, 1] \times \{\frac{1}{2}\}$. So $(\frac{1}{2}, 1] \times [0,1] \cap \mathbb{R}(\boldsymbol{\mu}) = (\frac{1}{2}, 1] \times [\frac{1}{2}, 1]$. This gives $F^1_{\boldsymbol{\mu}}(x) = \dfrac{1}{2}$; $F^2_{\boldsymbol{\mu}}(x) = 1$ for $\dfrac{1}{2} < x \le 1$. Thus, we see that the range $\mathbb{R}(\boldsymbol{\mu})$ consists of two rectangles $[0,\frac{1}{2}] \times [0,\frac{1}{2}]$ and $[\frac{1}{2}, 1] \times [\frac{1}{2}, 1]$ intersecting at $(\frac{1}{2}, \frac{1}{2})$ and, thus, is not convex. However, $\mathbb{R}(\boldsymbol{\mu})$ is compact and connected. Further, we have

$$F^1_{\boldsymbol{\mu}}(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{2} \\ \frac{1}{2}, & \frac{1}{2} < x \le 1 \end{cases} \quad \text{and} \quad F^2_{\boldsymbol{\mu}}(x) = \begin{cases} \frac{1}{2}, & 0 \le x < \frac{1}{2} \\ 1, & \frac{1}{2} \le x \le 1 . \end{cases}$$

The upper and lower bound of the range, viz. $F^1_{\boldsymbol{\mu}}$ and $F^2_{\boldsymbol{\mu}}$, are continuous on $[0,1] \sim \{\frac{1}{2}\}$. $F^1_{\boldsymbol{\mu}}$ is left continuous at $\frac{1}{2}$ and $F^2_{\boldsymbol{\mu}}$ is right continuous at $\frac{1}{2}$. Both $F^1_{\boldsymbol{\mu}}$ and $F^2_{\boldsymbol{\mu}}$ are increasing but have jump discontinuity at $\frac{1}{2}$. $F^1_{\boldsymbol{\mu}}$ cannot be the cumulative distribution function of a measure simply because it is not continuous on the right at $\frac{1}{2}$. Finally, $\mathbf{G}(\boldsymbol{\mu})$ consists of a one-dimensional closed non-convex set consisting of a horizontal and a vertical segment with common point $(\frac{1}{2}, \frac{1}{2})$.

(b) Now consider the measure pair $\boldsymbol{\nu} = (\nu_1, \nu_2)$ where $\nu = \mu_2, \nu_2 = \mu_1$.

Using part (a) above we obtain that Range $\mathbb{R}(\boldsymbol{\nu})$ = Range $\mathbb{R}(\boldsymbol{\mu})$. As a consequence $F^1_{\boldsymbol{\mu}} = F^1_{\boldsymbol{\nu}}, F^2_{\boldsymbol{\mu}} = F^2_{\boldsymbol{\nu}}$ and $\mathbf{G}(\boldsymbol{\nu}) = \mathbf{G}(\boldsymbol{\mu})$. Moreover, if $\mathbf{A} = \mathbb{R}(\boldsymbol{\mu})$ or $\mathbb{R}(\boldsymbol{\nu})$, then $C(\mathbf{A}) = D(\mathbf{A}) = \frac{1}{4}$.

5.4 Examples of two dimensional vector measures with totally disconnected range which is the graph of a continuous function

Let $\{r_n\}_{n=1}^{\infty}$ be any sequence of distinct points in the interval $[0,1]$.

(a) We now consider the measure pair $\boldsymbol{\mu} = (\mu_1, \mu_2)$ on $([0,1], \Sigma)$ given by

$$\mu_1 = \sum_{n=1}^{\infty} 2/3^n \delta_{r_n} \quad \text{and} \quad \mu_2 = \sum_{n=1}^{\infty} 1/2^n \delta_{r_n}.$$

We note that $\mu_2 \ll \mu_1$. We have

$$\mu_1(\Sigma) = C, \text{ the Cantor Ternary set and } \mu_2(\Sigma) = [0,1].$$

Every point 'x' of C can be written uniquely in the form $x = \sum\limits_{n=1}^{\infty} 2/3^n \varepsilon_n^x$ where $\varepsilon_n^x = 1$ or 0. Let $f : C \to [0,1]$ be the Lebesgue function (or Cantor function), that is, for each $x \in C$,

$$f(x) = \sum_{n=1}^{\infty} 1/2^n \varepsilon_n^x.$$

Then f is continuous and increasing on C.

Now let $x \in C$. Then for $E \in \Sigma$, $\mu_1 E = x$ if and only if

$$E \cap \{r_n : n \in \mathbb{N}\} = \{r_n : \varepsilon_n^x = 1\}.$$

So, $\mu_1 E = x$ gives $\mu_2 E = f(x)$. Thus the range $\mathbb{R}(\boldsymbol{\mu})$ is the graph of the function f and therefore totally disconnected. Further,

$$F^1_{\boldsymbol{\mu}} = F^2_{\boldsymbol{\mu}} = f.$$

Finally, $\mathbf{G}(\boldsymbol{\mu})$ consists of the singleton $\left\{(\frac{2}{3}, \frac{1}{2})\right\}$ expressing a bias towards the first person P_1.

(b) We now consider $\boldsymbol{\nu} = (\nu_1, \nu_2)$ where $\nu_1 = \mu_2$ and $\nu_2 = \mu_1$ with μ_1 and μ_2 as in (a) above. We first note that $\nu_1(\Sigma) = [0,1]$. We know that every $x \in [0,1]$ which is not a dyadic fraction, has a unique binary representation. Therefore part (a) gives

$$F^1_{\boldsymbol{\nu}}(x) = F^2_{\boldsymbol{\nu}}(x) = f^{-1}(x) \quad \text{for non-dyadic point '}x\text{'}.$$

For dyadic fraction $x = p/2^k$ (p odd and $1 \le p < 2^k$), we can write

$$x = \sum_{i=1}^{k} 1/2^i \varepsilon_i \quad \text{where} \quad \varepsilon_k = 1$$

and also $x = \sum_{i=1}^{k-1} 1/2^i \varepsilon_i + 0/2^k + \sum_{j=k+1}^{\infty} 1/2^j$. So, we have $F^1_{\boldsymbol{\nu}}(x) = \sum_{i=1}^{k-1} 2/3^i \varepsilon_i + \sum_{j=k+1}^{\infty} 2/3^j$ and $F^2_{\boldsymbol{\nu}}(x) = \sum_{i=1}^{k} 2/3^i \varepsilon_i$. Therefore $F^2_{\boldsymbol{\nu}}(x) = F^1_{\boldsymbol{\nu}}(x) + 1/3^k$. Also for $x_1 > x_2 > x_3$ we have $F^1_{\boldsymbol{\nu}}(x_1) \geq F^2_{\boldsymbol{\nu}}(x_2) \geq F^1_{\boldsymbol{\nu}}(x_2) \geq F^2_{\boldsymbol{\nu}}(x_3)$. Thus for any dyadic fraction x, $F^1_{\boldsymbol{\nu}}$ is right discontinuous at 'x' and $F^2_{\boldsymbol{\nu}}$ is left discontinuous at 'x'. The range $\mathbb{R}(\boldsymbol{\nu})$ =(the graph of $F^1_{\boldsymbol{\nu}}$) $\cup$ (the graph of $F^2_{\boldsymbol{\nu}}$), which is totally disconnected. Finally $\mathbf{G}(\boldsymbol{\nu})$ consists of the singleton $\{(\frac{1}{2}, \frac{2}{3})\}$ expressing a bias towards the second person P_2. Morever, if $\mathbf{A} = \mathbb{R}(\boldsymbol{\mu})$ or $\mathbb{R}(\boldsymbol{\nu})$, $C(\mathbf{A}) = \frac{1}{6}$ whereas $D(\mathbf{A}) \geq \frac{\sqrt{73}}{48}$. Clearly it is enough to prove the result for $\mathbf{A} = \mathbb{R}(\boldsymbol{\mu})$. The proof for this is elementary in nature but rather long. It is given in the appendix.

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6 Appendix regarding degree of convexity

Let $\boldsymbol{A} = \boldsymbol{R}(\boldsymbol{\mu})$, where $\boldsymbol{\mu}$ is the vector measure given in Example 5.4.

I. $C(\boldsymbol{A}) = \frac{1}{6}$. We begin by noting that $z_1 = (0, 0)$ and $z_2 = (1, 1) \in \mathbf{A}$ and $(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}(z_1 + z_2)$. Also $d\ ((\frac{1}{2}, \frac{1}{2}), \mathbf{A}) = \frac{1}{6}$. So $C(\mathbf{A}) = \sup\{d(\frac{x+y}{2}, \mathbf{A}) : x, y \in \mathbf{A}\} \geq \frac{1}{6}$. We now proceed to show that $C(\mathbf{A}) \leq \frac{1}{6}$. The set $\mathbf{B} = \{\sum_{i=1}^{n} \frac{2}{3^i}\varepsilon_i : n \in N \text{ where } \varepsilon_i = 0 \text{ or } 1\}$ is dense in C, therefore $R(f/\mathbf{B}) = \{(x, f(x)) : x \in \mathbf{B}\}$ is dense in $\mathbf{A}$. Let $x, y \in \mathbf{B}$, $x \neq y$. Then we may take

$$x = \sum_{i=1}^{n} \frac{2}{3^i}\varepsilon^x_i \quad \text{and} \quad y = \sum_{i=1}^{n} \frac{2}{3^i}\varepsilon^y_i,$$

where $\varepsilon^x_i = 0$ or $1, \varepsilon^y_i = 0$ or 1 for all i, we may assume that $\varepsilon^x_n + \varepsilon^y_n \neq 0$, otherwise we may take a smaller n. We have

$$f(x) = \sum_{i=1}^{n} \frac{1}{2^i}\varepsilon^x_i\,, \qquad f(y) = \sum_{i=1}^{n} \frac{1}{2^i}\varepsilon^y_i\,.$$

Also $u = \dfrac{x+y}{2} = \sum_{i=1}^{n} \dfrac{1}{3^i}(\varepsilon^x_i + \varepsilon^y_i)$ and $v = \dfrac{f(x)+f(y)}{2} = \sum_{i=1}^{n} \dfrac{1}{2^{i+1}}(\varepsilon^x_i + \varepsilon^y_i)\,.$

We now consider two cases, when $\varepsilon^x_1 + \varepsilon^y_1 = 1$ and when $\varepsilon^x_1 + \varepsilon^y_1 = 0$ or 2.

Case I. $\varepsilon^x_1 + \varepsilon^y_1 = 1$. Then $\frac{1}{3} \leq u < \frac{2}{3}$.

(i) Now suppose $\frac{1}{3} \leq u \leq \frac{7}{18} = \frac{1}{3} + \frac{0}{3^2} + \sum_{i=3}^{\infty} \frac{1}{3^i}$. Then $\varepsilon^x_2 + \varepsilon^y_2 = 0$, and, therefore, $\frac{1}{4} \le v < \frac{1}{2}$. We note that $\frac{1}{4} = f(\frac{2}{3^2})$ and $\frac{1}{2} = f(\frac{1}{3})$ and f has range $[0,1]$ and is continuous and increasing on C. Therefore, there exists $w \in C$ such that $\frac{2}{9} \le w \le \frac{1}{3}$ and $v = f(w)$. Then $0 \le u - w \le \dfrac{7}{18} - \dfrac{2}{9} = \dfrac{1}{6}$. Thus $d((u,v),(w,f(w))) \le \frac{1}{6}$ and $(w,f(w)) \in \mathbf{A}$.

(ii) Now suppose

$$\frac{1}{3} + \frac{0}{3^2} + \sum_{i=3}^{\infty} \frac{1}{3^i} = \frac{7}{18} < u \le \frac{1}{2} = \sum_{i=1}^{\infty} \frac{1}{3^i}.$$

Then two subcases arise

(a) $\varepsilon_2^x + \varepsilon_2^y = 0$, and

(b) $\varepsilon_2^x + \varepsilon_2^y = 1$.

(a) As $\varepsilon_2^x + \varepsilon_2^y = 0$, $\frac{7}{18} < u$ gives $\varepsilon_i^x + \varepsilon_i^y = 2$ for some $i \ge 3$. Let k be the smallest integer such that $\varepsilon_k^x + \varepsilon_k^y = 2$, then $\varepsilon_j^x + \varepsilon_j^y = 1$ for $2 < j < k$. So we have

$$\begin{aligned} u &= \frac{1}{3} + \frac{0}{3^2} + \sum_{i=3}^{k-1} \frac{1}{3^i} + \frac{2}{3^k} + \sum_{i=k+1}^{n} \frac{1}{3^i}(\varepsilon_i^x + \varepsilon_i^y) \\ &< \frac{1}{3} + \sum_{i=3}^{\infty} \frac{2}{3^i} = \frac{1}{3} + \frac{1}{3^2}, \end{aligned}$$

and

$$\begin{aligned} v &= \sum_{i=1}^{n} \frac{1}{2^{i+1}}(\varepsilon_i^x + \varepsilon_i^y) \\ &= \frac{1}{2^2} + \frac{0}{2^3} + \sum_{i=3}^{k-1} \frac{1}{2^{i+1}} + \frac{2}{2^{k+1}} + \sum_{i=k+1}^{n} \frac{1}{2^{i+1}}(\varepsilon_i^x + \varepsilon_i^y) \\ &= \frac{1}{2^2} + \frac{1}{2^3} + \sum_{i=k+1}^{n} \frac{1}{2^{i+1}}(\varepsilon_i^x + \varepsilon_i^y) \\ &\le \frac{1}{4} + \frac{1}{8} + \frac{1}{2^k} \le \frac{1}{2}. \end{aligned}$$

So $\frac{1}{4} + \frac{1}{8} \le v \le \frac{1}{2}$. Now $\frac{1}{4} + \frac{1}{8} = f(\frac{2}{3^2} + \frac{2}{3^3})$ and $\frac{1}{2} = f(\frac{1}{3})$ and f has range $[0,1]$ and is continuous and increasing on C. Thus, there exists $w \in C$ with $\frac{1}{3} \ge w \ge \frac{2}{3^2} + \frac{2}{3^3}$ such that $f(w) = v$. Also $\frac{1}{3} < u < \frac{1}{3} + \frac{1}{3^2} = \frac{4}{9}$. Therefore we have $0 < u - w < \frac{4}{9} - \frac{2}{9} - \frac{2}{27} = \frac{4}{27} < \frac{1}{6}$.

(b) Because $\varepsilon_2^x + \varepsilon_2^y = 1$, $u \le 1/2$ gives either $\varepsilon_i^x + \varepsilon_i^y \le 1$ for all i or there exists $k \ge 3$ such that $\varepsilon_k^x + \varepsilon_k^y = 2$. In the subcase when $\varepsilon_i^x + \varepsilon_i^y \le 1$ for all i, we have $(\frac{2}{3}u) \in \mathbf{B}$. Also

$$v=\sum_{i=1}^{n}\frac{1}{2^{i+1}}(\varepsilon_i^x+\varepsilon_i^y)=f(\sum_{i=1}^{n}\frac{2}{3^{i+1}}(\varepsilon_i^x+\varepsilon_i^y))=f((\frac{2}{3})u)\,.$$

The distance of u from $(\frac{2}{3})u=(\frac{1}{3})u<\frac{1}{6}$.

We now come to the subcase when there exists a least 'k' such that $\varepsilon_k^x+\varepsilon_k^y=2$. Since $u\le\frac{1}{2}$, we have $k>3$ and there exists a largest 'p' with $3\le p<k$ such that $\varepsilon_p^x+\varepsilon_p^y=0$. Therefore we have

$$u=\sum_{i=1}^{n}\frac{1}{3^i}(\varepsilon_i^x+\varepsilon_i^y)=\frac{1}{3}+\frac{1}{3^2}+\sum_{t=3}^{p-1}\frac{1}{3^t}\varepsilon_t+\frac{0}{3^p}+\sum_{i=p+1}^{k-1}\frac{1}{3^i}+\frac{2}{3^k}+\sum_{s=k+1}^{n}\frac{1}{3^s}\varepsilon_s$$

where $\varepsilon_t=0$ or 1 and $\varepsilon_s=0,1$ or 2. Therefore

$$\frac{1}{3}+\frac{1}{3^2}<u<\frac{1}{3}+\frac{1}{3^2}+\sum_{t=3}^{p-1}\frac{1}{3^t}\varepsilon_t+\sum_{i=p+1}^{\infty}\frac{2}{3^i}=\frac{1}{3}+\frac{1}{3^2}+\sum_{t=3}^{p-1}\frac{1}{3^t}\varepsilon_t+\frac{1}{3^p},$$

and

$$\begin{aligned}v&=\textstyle\sum_{i=1}^{n}\frac{1}{2^{i+1}}(\varepsilon_i^x+\varepsilon_i^y)\\&=\tfrac{1}{2^2}+\tfrac{1}{2^3}+\sum_{t=3}^{p-1}\tfrac{1}{2^{t+1}}\varepsilon_t+\tfrac{0}{2^{p+1}}+\sum_{i=p+1}^{k-1}\tfrac{1}{2^{i+1}}+\tfrac{2}{2^{k+1}}+\sum_{s=k+1}^{n}\tfrac{1}{2^{s+1}}\varepsilon_s\\&=\tfrac{1}{2^2}+\tfrac{1}{2^3}+\sum_{t=3}^{p-1}\tfrac{1}{2^{t+1}}\varepsilon_t+1/2^{p+1}+\sum_{s=k+1}^{n}\tfrac{1}{2^{s+1}}\varepsilon_s\,.\end{aligned}$$

Therefore $\frac{1}{2}\ge v\ge\frac{1}{2^2}+\frac{1}{2^3}+\sum_{t=3}^{p-1}\frac{1}{2^{t+1}}\varepsilon_t+\frac{1}{2^{p+1}}$. Now $\frac{1}{2}=f(\frac{1}{3})$ and $\frac{1}{2^2}+\frac{1}{2^3}+\sum_{t=3}^{p-1}\frac{1}{2^{t+1}}\varepsilon_t+\frac{1}{2^{p+1}}=f(\frac{2}{3^2}+\frac{2}{3^3}+\sum_{t=3}^{p-1}\frac{2}{3^{t+1}}\varepsilon_t+\frac{2}{3^{p+1}})$. Also, f has range $[0,1]$ and is continuous and increasing on C. Therefore there exists $w\in C$ such that

$$\frac{2}{3^2}+\frac{2}{3^3}+\sum_{t=3}^{p-1}\frac{2}{3^{t+1}}\varepsilon_t+\frac{2}{3^{p+1}}\le w\le\frac{1}{3}\text{ and }f(w)=v.$$

So we have

$$\begin{aligned}0<u-w&\le\frac{1}{3}+\frac{1}{3^2}+\sum_{t=3}^{p-1}\frac{1}{3^t}\varepsilon_t+\frac{1}{3^p}-\frac{2}{3^2}-\frac{2}{3^3}-\sum_{t=3}^{p-1}\frac{2}{3^{t+1}}\varepsilon_t-\frac{2}{3^{p+1}}\\&=\frac{1}{3^2}+\frac{1}{3^3}+\sum_{t=3}^{p-1}\frac{1}{3^{t+1}}\varepsilon_t+\frac{1}{3^{p+1}}<\frac{1}{3}\sum_{t=1}^{\infty}\frac{1}{3^t}=\frac{1}{6}\,.\end{aligned}$$

(iii) Now, suppose $\frac{1}{2}=\sum_{i=1}^{\infty}\frac{1}{3^i}<u<\frac{2}{3}=\frac{1}{3}+\sum_{i=2}^{\infty}\frac{2}{3^i}$. Then $n>1$ and there exists 'k' with $1\le k<n$ such that $\varepsilon_i^x+\varepsilon_i^y=1$ for $1\le i\le k$ and $\varepsilon_{k+1}^x+\varepsilon_{k+1}^y=2$. Again we consider three subcases: (a) $k+1=n$, (b) $k+2=n, n\ge3$, and (c) $k+2<n, n\ge4$.

(a) As $k+1=n$, we have $u = \sum\limits_{i=1}^{n-1} \frac{1}{3^i} + \frac{2}{3^n} = \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{3^n}$. and $v = \sum\limits_{i=1}^{n-1} \frac{1}{2^{i+1}} + \frac{2}{2^{n+1}} = \frac{1}{2}$. Let $w = \frac{2}{3}$, then $w \in \mathbf{B}$ and $f(w) = \frac{1}{2} = v$. Also $0 < w - u = \frac{2}{3} - \frac{1}{2} - \frac{1}{2}\cdot\frac{1}{3^n} = \frac{1}{6} - \frac{1}{2}\cdot\frac{1}{3^n} < \frac{1}{6}$.

(b) As $k+2=n$, we have

$$v = \sum_{i=1}^{k} \frac{1}{2^{i+1}} + \frac{2}{2^{k+2}} + \frac{1}{2^{n+1}}(\varepsilon_n^x + \varepsilon_n^y) = \frac{1}{2} + \frac{1}{2^{n+1}}(\varepsilon_n^x + \varepsilon_n^y)$$

If $\varepsilon_n^x + \varepsilon_n^y = 1$, put $w = \frac{2}{3} + \frac{1}{3^{n+1}}$ which gives $w \in C$ and $f(w) = \frac{1}{2} + \frac{1}{2^{n+1}} = v$. Also $u = \sum\limits_{i=1}^{n-2} \frac{1}{3^i} + \frac{2}{3^{n-1}} + \frac{1}{3^n} = \frac{1}{2} - \frac{1}{2}\cdot\frac{1}{3^n} + \frac{1}{3^{n-1}}$. Therefore

$0 \le w - u = \frac{2}{3} + \frac{1}{3^{n+1}} - \frac{1}{2} - \frac{1}{3^{n-1}} + \frac{1}{2}\cdot\frac{1}{3^n} = \frac{1}{6} - \frac{1}{2}\cdot\frac{13}{3^{n+1}} < \frac{1}{6}$.

If $\varepsilon_n^x + \varepsilon_n^y = 2$, put $w = \frac{2}{3} + \frac{1}{3^n}$, which gives $w \in C$ and $f(w) = \frac{1}{2} + \frac{1}{2^n} = v$. Also $u = \sum_{i=1}^{n-2} \frac{1}{3^i} + \frac{2}{3^{n-1}} + \frac{2}{3^n} = \frac{1}{2} + \frac{7}{2}\cdot\frac{1}{3^n}$. Therefore

$$0 \le w - u = \frac{2}{3} + \frac{1}{3^n} - \frac{1}{2} - \frac{7}{2}\cdot\frac{1}{3^n} = \frac{1}{6} - \frac{1}{2}\cdot\frac{5}{3^n} < \frac{1}{6}.$$

(c) Now $k+2 < n$. So

$$u = \sum_{i=1}^{k} \frac{1}{3^i} + \frac{2}{3^{k+1}} + \sum_{p=k+2}^{n} \frac{1}{3^p}(\varepsilon_p^x + \varepsilon_p^y) = \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{3^{k+1}} + \sum_{p=k+2}^{n} \frac{1}{3^p}(\varepsilon_p^x + \varepsilon_p^y),$$

and

$$v = \sum_{i=1}^{k} \frac{1}{2^{i+1}} + \frac{2}{2^{k+2}} + \sum_{p=k+2}^{n} \frac{1}{2^{p+1}}(\varepsilon_p^x + \varepsilon_p^y) = \frac{1}{2} + \sum_{p=k+2}^{n} \frac{1}{2^{p+1}}(\varepsilon_p^x + \varepsilon_p^y).$$

Put $\delta_i = 0$ if $\varepsilon_i^x + \varepsilon_i^y = 0$, $\delta_i = 1$ if $\varepsilon_i^x + \varepsilon_i^y = 1$ or 2. Then $\delta_i \le \varepsilon_i^x + \varepsilon_i^y \le 2\delta_i$ for all i and $\delta_n = 1$. Further, $\frac{1}{2} < v \le \frac{1}{2} + \sum\limits_{p=k+2}^{n} \frac{1}{2^{p+1}} 2\delta_p = \frac{1}{2} + \sum\limits_{p=k+2}^{n} \frac{1}{2^p}\delta_p$. Also $f(\frac{2}{3}) = \frac{1}{2}$ and $f(\frac{2}{3} + \sum\limits_{p=k+2}^{n-1} \frac{2}{3^p}\delta_p + \frac{1}{3^n}) = \frac{1}{2} + \sum\limits_{p=k+2}^{n} \frac{1}{2^p}\delta_p$. As f has range $[0,1]$ and is continuous and increasing on C, there exists $w \in C$ such that $f(w) = v$ and $\frac{2}{3} \le w \le \frac{2}{3} + \sum\limits_{p=k+2}^{n-1} \frac{2}{3^p}\delta_p + \frac{1}{3^n}$. Hence,

$$\begin{aligned} 0 \le w - u &= \tfrac{2}{3} + \sum_{p=k+2}^{n-1} \tfrac{2}{3^p}\delta_p + \tfrac{1}{3^n} - \tfrac{1}{2} - \tfrac{1}{2}\cdot\tfrac{1}{3^{k+1}} - \sum_{p=k+2}^{n} \tfrac{1}{3^p}(\varepsilon_p^x + \varepsilon_p^y) \\ &= \tfrac{1}{6} + \sum_{p=k+2}^{n-1} \tfrac{1}{3^p}(2\delta_p - (\varepsilon_p^x + \varepsilon_p^y)) + \tfrac{1}{3^n}(1 - (\varepsilon_n^x + \varepsilon_n^y)) - \tfrac{1}{2}\cdot\tfrac{1}{3^{k+1}} \\ &\le \tfrac{1}{6} + \sum_{p=k+2}^{n} \tfrac{1}{3^p} - \tfrac{1}{2}\cdot\tfrac{1}{3^{k+1}} < \tfrac{1}{6}. \end{aligned}$$

Case II. When $\varepsilon_1^x + \varepsilon_1^y = 0$ or 2. Two possibilities arise.

(i) $\varepsilon_2^x + \varepsilon_2^y = 0$ or 2. We put $\delta_i = 0$ if $\varepsilon_i^x + \varepsilon_i^y = 0$ or 1, $\quad \delta_i = 1$ if $\varepsilon_i^x + \varepsilon_i^y = 2$. We take $w = \sum_{i=1}^{n} \frac{2}{3^i}\delta_i$, then $w \in \mathbf{B}$, $\boldsymbol{w} \leq u$ and $f(w) = \sum_{i=1}^{n} \frac{1}{2^i}\delta_i$. Therefore

$$0 \leq u - w = \sum_{i=1}^{n} \frac{1}{3^i}(\varepsilon_i^x + \varepsilon_i^y) - \sum_{i=1}^{n} \frac{2}{3^i}\delta_i = \sum_{i=3}^{n} \frac{1}{3^i}(\varepsilon_i^x + \varepsilon_i^y - 2\delta_i) < \sum_{i=3}^{\infty} \frac{1}{3^i} = \frac{1}{18},$$

and

$$\begin{aligned} 0 \leq v - f(w) &= \sum_{i=1}^{n} \frac{1}{2^{i+1}}(\varepsilon_i^x + \varepsilon_i^y) - \sum_{i=1}^{n} \frac{1}{2^i}\delta_i \\ &= \sum_{i=3}^{n} \frac{1}{2^{i+1}}(\varepsilon_i^x + \varepsilon_i^y - 2\delta_i) < \sum_{i=3}^{\infty} \frac{1}{2^{i+1}} = \frac{1}{8}. \end{aligned}$$

Hence $d((u,v),(w,f(w))) \leq (\frac{1}{18^2} + \frac{1}{8^2})^{\frac{1}{2}} = \frac{\sqrt{97}}{72} < \frac{1}{6}$.

(ii) $\varepsilon_2^x + \varepsilon_2^y = 1$. We consider two subcases (a) $\varepsilon_1^x + \varepsilon_1^y = 0$, and (b) $\varepsilon_1^x + \varepsilon_1^y = 2$.

(a) When $\varepsilon_1^x + \varepsilon_1^y = 0$, we have $\frac{1}{3^2} \leq u < \frac{2}{3^2}$, and $\frac{1}{2^3} \leq v \leq \frac{1}{2^3} + \sum_{i=3}^{n} \frac{2}{2^{i+1}} = \frac{1}{2^3} + \sum_{i=3}^{n} \frac{1}{2^i} \leq \frac{1}{2^2} + \frac{1}{2^3}$. Also, $\frac{1}{2^2} + \frac{1}{2^3} = f(\frac{2}{3^2} + \frac{1}{3^3})$, $\frac{1}{2^3} = f(\frac{2}{3^3})$. Since f has range $[0,1]$ and is continuous and increasing on C, there exists $w \in C$ such that $v = f(w)$ and $\frac{2}{3^3} \leq w \leq \frac{2}{3^2} + \frac{1}{3^3}$. Therefore, we have

$$|w - u| \leq \max\{\frac{2}{3^2} - \frac{2}{3^3}, \frac{2}{3^2} + \frac{1}{3^3} - \frac{1}{3^2}\} = \frac{4}{27} < \frac{1}{6}.$$

(b) Now in this second subcase when $\varepsilon_1^x + \varepsilon_1^y = 2$, we have $\frac{2}{3} + \frac{1}{3^2} \leq u \leq \frac{2}{3} + \frac{2}{3^2}$ and $\frac{1}{2} + \frac{1}{2^3} \leq v \leq \frac{1}{2} + \frac{1}{2^3} + \sum_{i=3}^{n} \frac{2}{2^{i+1}} \leq \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}$. Since $\frac{1}{2} + \frac{1}{2^3} = f(\frac{2}{3} + \frac{2}{3^3})$ and $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} = f(\frac{2}{3} + \frac{2}{3^2} + \frac{1}{3^3})$ and f has range $[0,1]$ and is continuous and increasing on C, there exists $w \in C$ such that $f(w) = v$ and $\frac{2}{3} + \frac{2}{3^3} \leq w \leq \frac{2}{3} + \frac{2}{3^2} + \frac{1}{3^3}$. Therefore, we have

$$|w - u| \leq \max\left\{\frac{2}{3} + \frac{2}{3^2} - \frac{2}{3} - \frac{2}{3^3}, \frac{2}{3} + \frac{2}{3^2} + \frac{1}{3^3} - \frac{2}{3} - \frac{1}{3^2}\right\} = \frac{4}{27} < \frac{1}{6}.$$

Finally, we have $d((\frac{x+y}{2}, \frac{f(x)+f(y)}{2}), \mathbf{A}) \leq \frac{1}{6}$. As $x, y \in \mathbf{B}$, $x \neq y$ are arbitrary and $\mathbb{R}(f/\mathbf{B})$ is dense in $\mathbb{R}(\boldsymbol{\mu})$, we have $\sup\{d(\frac{1}{2}(z_1 + z_2), \mathbf{A}) : z_1, z_2 \in \mathbf{A}\} \leq \frac{1}{6}$. Hence $C(\mathbf{A}) \leq \frac{1}{6}$.

II. $D(A) \geq \frac{\sqrt{73}}{48}$

The point $P = (\frac{1}{2}, \frac{3}{8}) = \frac{1}{4}(0,0) + \frac{3}{4}(\frac{2}{3}, \frac{1}{2}) \in \mathrm{Co}(\mathbf{A})$. We shall show that $d(P, \mathbf{A}) \geq \frac{\sqrt{73}}{48} > \frac{1}{6}$. Since $\mathbb{R}(f/\mathbf{B})$ is dense in $\mathbf{A}$, we have $d(P, \mathbf{A}) = d(P, \mathbb{R}(f/\mathbf{B}))$. Let us write $P = (\frac{1}{2}, \frac{3}{8}) = (\frac{1}{3} + \frac{1}{6}, \frac{1}{4} + \frac{1}{8})$. Take $x \in \mathbf{B}$. We consider three cases

Case 1. When $f(x) \leq \frac{3}{8} = \frac{1}{4} + \frac{1}{8}$. Then $x \leq \frac{2}{3^2} + \frac{2}{3^3} = \frac{8}{27}$. So $d(P, (x, f(x))) \geq \frac{1}{3} + \frac{1}{6} - \frac{8}{27} = \frac{11}{54}$.

Case 2. When $\frac{3}{8} < f(x) < \frac{1}{2}$. We may take $x = \sum\limits_{i=1}^{n} \frac{2}{3^i}\varepsilon_i^x$ with $\varepsilon_i^x = 0$ or 1 and $\varepsilon_x^n \neq 0$. As $\frac{1}{2^2} + \frac{1}{2^3} < f(x)$, we have $n \geq 4$,

$$f(x) = \frac{0}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \sum_{i=4}^{n} \frac{1}{2^i}\varepsilon_i^x \text{ and } x = \frac{0}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \sum_{i=4}^{n} \frac{2}{3^i}\varepsilon_i^x .$$

So

$$\begin{aligned} & d(P, (x, f(x))) \\ =& \left\{ \left(\tfrac{1}{3} + \tfrac{1}{6} - \left(\tfrac{2}{3^2} + \tfrac{2}{3^3} + \sum_{i=4}^{n} \tfrac{2}{3^i}\varepsilon_i^x \right) \right)^2 + \left(\tfrac{3}{8} - \left(\tfrac{1}{2^2} + \tfrac{1}{2^3} + \sum_{i=4}^{n} \tfrac{1}{2^i}\varepsilon_i^x \right) \right)^2 \right\}^{\frac{1}{2}} \\ =& \left\{ \left(\tfrac{1}{6} + \sum_{i=4}^{n} \tfrac{2}{3^i}(1 - \varepsilon_i^x) + \tfrac{1}{3^n} \right)^2 + \left(\sum_{i=4}^{n} \tfrac{1}{2^i}\varepsilon_i^x \right)^2 \right\}^{\frac{1}{2}}. \end{aligned}$$

Now either $\varepsilon_4^x = 0$ or 1. If $\varepsilon_4^x = 1$, then

$$d(P, (x, f(x))) \geq \left\{ \left(\frac{1}{6} \right)^2 + \left(\frac{1}{2^4} \right)^2 \right\}^{\frac{1}{2}} = \frac{\sqrt{73}}{48} .$$

If $\varepsilon_4^x = 0$. Then

$$d(P, (x, f(x))) > \left\{ \left(\frac{1}{6} + \frac{2}{3^4} \right)^2 \right\}^{\frac{1}{2}} = \frac{31}{162} > \frac{\sqrt{73}}{48} .$$

Case 3. When $f(x) > \frac{1}{2}$. Then $x > \frac{2}{3}$. We have

$$d(P, (x, f(x))) \geq \left\{ \left(\frac{1}{3} + \frac{1}{6} - \frac{2}{3} \right)^2 + \left(\frac{1}{4} + \frac{1}{8} - \frac{1}{2} \right)^2 \right\}^{\frac{1}{2}} = \frac{5}{24} .$$

Therefore for any $x \in \mathbf{B}$

$$d(P, (x, f(x))) \geq \min\{ \frac{11}{54}, \frac{\sqrt{73}}{48}, \frac{5}{24} \} = \frac{\sqrt{73}}{48} .$$

So $d(P, \mathbf{A}) \geq \frac{\sqrt{73}}{48}$. Thus, $D(A) = \sup\{ d(x, \mathbf{A}) : x \in \mathrm{Co}(\mathbf{A}) \} \geq \frac{\sqrt{73}}{48}$.

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