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THE STABILITY OF RETICULAR MAP GERMS

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Abstract

The purpose of this paper is to introduce notations of r-reticular map germs on $\mathbb{R}^n$ and investigate their stability. The main results are proofs of stability theorems, which is the similar to Mather's stable theorems for C^∞-maps.

1 Introduction

In the Mather's stable theory for C^∞ map-germ, in order to prove stable criteria, J. Mather gave various equivalent notations of stability of map-germs. They are Stable map; Transverse stability; Infinitesimally stable; Homotopically stable; Stability under deformation, and he proved that all notations are equivalent (see [3], pp. 111-142).

We propose in this paper to generalize Mather's theory of stable map-germ and to prove stable theorems similar in Mather's stable theorem for C^∞ map-germ by unfolding theory.

The paper contains two sections: The first section deals with the basic notation and in Section 2, we prove some results concerning the stability of r-reticular map-germ in $\mathbb{R}^n$. The main results of this paper are theorems 3.1, 3.4 and 3.5.

Key words: r-reticular map germs, Mather's stable theorems, r-reticular manifold.
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2 Preliminaries

We denote by $\varepsilon(n,p)$ the set of germs of differentiable maps at *zero* on $\mathbb{R}^n$ to $\mathbb{R}^p$. We write $\varepsilon(n)$ instead $\varepsilon(n,1)$.

Fix $r \in \mathbb{N}, 0 \leq r \leq n$, let X_i denote a germ of the set $\{(x_1, x_2, ..., x_n) \in \mathbb{R}^n | x_i = 0\}$, and let $P(I_r)$ denote the family of all subsets of the set $I_r = \{1, 2, ..., r\}$.

The collection $\underline{X} = (X_\sigma)_{\sigma \in P(I_r)}$ where $X_\sigma = \cap_{i\in\sigma} X_i$, is called a germ of r-reticular manifold. Denote by $\text{Diff}^r(n) = \{\Phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0) | \Phi$ is a diffeomorphism germ at $zero$, such that $\Phi(X_\sigma) = X_\sigma, \sigma \in P(I_r)\}$ and $Y \equiv \mathbb{R}^m$.

2.1 Definition Let $f : \underline{X} \to Y$ be a function from X into Y. Denote by $\underline{f}$ the collection $(f_\sigma)_{\sigma\in P(I)}$, where $f_\sigma = f|_{X_\sigma}$. $\underline{f}$ is called a r-reticular map-germ, we write $\underline{f} : \underline{X} \to Y$.

We say that two germs $\underline{f}_1, \underline{f}_2 : (\underline{X}, 0) \to (Y, 0)$ are reticularly equivalent if there exists $\phi \in \text{Diff}^r(n)$ such that $\underline{f}_1 = \underline{f}_2 \circ \phi$.

Example 1 Let $X = I\!R^3,\ Y = I\!R,\ I = \{1, 2\}$ then

$P(I) = \{\{\emptyset\}, \{1\}, \{2\}, \{1, 2\}\}$. Let $f : \underline{X} \to I\!R,\ (x, y, z) \mapsto x^2 + 2xy + yz$. Then we have

if $\sigma_1 = \sigma = \phi \in P(I)$, then $f_{\sigma_1} : I\!R^3 \to I\!R,\ f_{\sigma_1}(x, y, z) = x^2 + 2xy + yz$;
if $\sigma_2 = \sigma = \{1\} \in P(I)$, then $f_{\sigma_2} : X_{\sigma_2} \to I\!R,\ f_{\sigma_2}(0, y, z) = yz$;
if $\sigma_3 = \sigma = \{2\} \in P(I)$, then $f_{\sigma_3} : X_{\sigma_3} \to I\!R,\ f_{\sigma_3}(x, 0, z) = x^2$;
if $\sigma_4 = \sigma = \{1, 2\} \in P(I)$, then $f_{\sigma_4} : X_{\sigma_4} \to I\!R,\ f_{\sigma_4}(0, 0, z) = 0$.

It is easily seen that $\underline{f} = \{f_{\sigma_1}, f_{\sigma_2}, f_{\sigma_3}, f_{\sigma_4}\}$, where $X_{\sigma_1} = X_{\{\emptyset\}} = I\!R^3,\ X_{\sigma_2} = X_{\{1\}} = \{(0, y, z) \in I\!R^3\},\ X_{\sigma_3} = X_{\{2\}} = \{(x, 0, z) \in I\!R^3\},\ X_{\sigma_4} = X_{\{1,2\}} = X_1 \cap X_2 = \{(0, 0, z) \in I\!R^3\}$.

Example 2 Let $X = I\!R^3,\ Y = I\!R,\ I = \{1, 2\},\ f : \underline{X} \to I\!R,\ (x, y, z) \mapsto x^2 + 2xy + yz$ and $g : \underline{X} \to I\!R,\ (x, y, z) \mapsto \dfrac{x^2}{4} + xy + yz$. Then $\underline{f}, \underline{g}$ are reticularly equivalent. Indeed, taking $\phi : (I\!R^3, 0) \to (I\!R^3, 0),\ (x, y, z) \mapsto (2x, y, z)$, we have $\phi(X_{\sigma_i}) = X_{\sigma_i},\ i = 1, 2, 3, 4$ and $\underline{f} = \underline{g} \circ \phi$.

Remark (a) By a change of basis in $I\!R^n$, each element ϕ of $\text{Diff}^r(n)$ is of the form $\phi(x) = (x_1\phi_1(x), \ldots, x_r\phi_r(x), \phi_{r+1}(x), \ldots, \phi_n(x))$ where ϕ_i are germs of smooth functions on $I\!R^n$ and $Jac(\phi) \neq 0$.

(b) Let U be an open subset of $I\!R^q$. Then $\underline{X} \times U = (X_\sigma \times U)|_{\sigma\in P(I)}$ is also a germ of a reticular manifold.

2.2 Definition Let $f : \underline{X} \to Y$ be a r-reticular map-germ, and U be an open subset of $I\!R^q$ (called the base of the unfolding). We say that a reticular map-germ $F : \underline{X} \times U \ni (x, u) \to (F(x, u), u) \in Y \times U$ is an unfolding of $\underline{f}$ if $F(x, 0) = \underline{f}(x)$.

We say that two unfolding F_1 and F_2 are equivalent if there exist $\phi \in \text{Diff}^r(n + q),\ G \in \varepsilon(m + q, m + q)$ (where $\dim Y = m$) such that ϕ (resp. G)

is an unfolding of the identity of $\underline{X}$ (resp. Y) and the following diagram is commutative

$$\begin{array}{ccc} \underline{X}\times U & \xrightarrow{F_1} & Y\times U \\ \Big\downarrow \phi & & \Big\downarrow G \\ \underline{X}\times U & \xrightarrow{F_2} & Y\times U \end{array}$$

Let F be as above and let $h : (U', 0) \to (U, 0)$ be a germ of a diffeomorphism and $h(u') \in U$. Then the map germ h^*F defined by $(h^*F)(x, u') = (F(x, h(u')), u')$ is called the unfolding derived from the unfolding $F : \underline{X}\times U \to Y \times U$ of $\underline{f}$ by the change of basis h.

We say that an unfolding F of $\underline{f}$ is versal if any unfolding of $\underline{f}$ is reticularly equivalent to an unfolding induced from F by a change of basis.

A reticular map germ is stable if any unfolding $F : \underline{X} \times U \to Y \times U$ is trivial, i.e., it is reticularly equivalent to the constant unfolding

$$\underline{X} \times U \ni (x, u) \to (\underline{f}(x), u) \in (Y \times U).$$

Let $f : \underline{X} \to Y$ be a smooth map-germ, we say that $\underline{f}$ is infinitesimally stable if for any $g \in \varepsilon(n, m)$, where $\varepsilon(n, m)$ is the ring of smooth functions from $I\!R^n$ into Y, there exist $h_i \in \varepsilon(n)$ and $k_i \in \varepsilon(m)$ such that

$$g(x) = \sum_{i=1}^{r} x_i \frac{\partial f}{\partial x_i} h_i(x) + \sum_{i=r+1}^{n} \frac{\partial f}{\partial x_i} h_i(x) + \sum_{j=1}^{m} k_j(f(x))e_j.$$

Where $(e_1, \ldots, e_m)$ is canonical basis of $\varepsilon(n, m)$. We write

$$T_r f = \varepsilon(n)\{x_1 \frac{\partial f}{\partial x_1}, \ldots, x_r \frac{\partial f}{\partial x_r}, \frac{\partial f}{\partial x_{r+1}}, \ldots \frac{\partial f}{\partial x_n}\} + f^*\varepsilon(m)(e_1, \ldots, e_m)$$

where $f^*\varepsilon(m)$ is the subring of $\varepsilon(n)$ defined by $f^*\varepsilon(n) = \{k_o f : k \in \varepsilon(m)\}\varepsilon(n)$.

The number $\mathrm{Cod}_r(f) = \dim_{I\!R}(\varepsilon(n, m)/T_r f)$ is called the reticular codimention of $\underline{f}$.

Let F be an unfolding of $\underline{f}$ with base U. We say that F is infinitesimally versal if

$$T_r f + I\!R\{\dot{F}_1, \dot{F}_2, \ldots, \dot{F}_q\} = \varepsilon(n, m) \tag{1}$$

where $F(x, u) = (F(x, u), u)$, $\dot{F}_i\ (x) = \dfrac{\partial F}{\partial u_i}(x, 0)$ and $I\!R\{\dot{F}_1, \dot{F}_2, \ldots, \dot{F}_q\}$ is the vector subspace of $\varepsilon(n, m)$ generated by $\{\dot{F}_1, \dot{F}_2, \ldots, \dot{F}_q\}$.

Remark We can prove that $\underline{f}$ is infinitesimally stable if and only if any infinitesimal unfolding $\underline{f} + ta$ is trivial (i.e $a \in T_r f$)

3 Some Results

3.1 Theorem *A smooth map-germ f has a versal unfolding if and only if $Cod_r(f)$ is finite.*

Proof Assume that $\mathrm{Cod}_r(f) = l$, and $g_1, \ldots, g_l$ form a basis of $\varepsilon(n,m)/T_r f$. Then the map-germ defined by

$$F(x,u) = (\underline{f}(x) + \sum_{i=1}^{l} u_i g_i(x), u)$$

is infinitesimally versal unfolding (since it satisfies (1)). In order to proceed, now we are interested in the case, all map-germs are holomorphic. In this case, instead of $\varepsilon(n)$, we denote by $O(n), O(p), O(n,p)$ the rings of germs of holomorphic functions on the germs X, Y and of germs of holomorphic maps of the germ X to the germ Y respectively, where $n = \dim X, p = \dim Y$. We need the following theorem.

3.2 Theorem *Any infinitesimally versal deformation is versal.*

The proof of this theorem is similar to that of theorem 3.3 in [5], if we know a generalization of the flow-box theorem. Hence it is enough to formulate this generalization and prove it.

3.3 Flow-box Theorem (in a generalized form) *Let $v(x) = \sum_{i=1}^{n} \xi_i(x) \frac{\partial}{\partial x_i}$ be a germ of a vector-field at 0, where $\xi_i \in \varepsilon(n)$ and $\xi_i(x) = x_i \widetilde{\xi_i}(x),\ i = 1, \ldots, r$. Let h_t be the flow generated by v. Then, in a small neighbourhood of $t = 0$, h_t is a reticular analysis map-germ.*

Proof a) By the usual flow-box theorem, h_t is a analysis map-germ. The proof can be found in [V.Arnold, Ordinary Dif and only if erential Equations, Moscow 1974]. Observe that this is the holomorphic version of the usual flow-box theorem.

b) Let h_t be of the form: $h_t(x) = (h_t^1(x), \ldots, h_t^n(x))$, it is enough to prove that,

(3) $h_t^i|_{x_i=0} = 0$ for $i = 1, \ldots, r$.

In oder to prove (3), we show that $\dfrac{d^s h_t^i}{dt^s}|_{t=0} = 0$ on $x_i = 0$ for all s (since by analyticity of the mapping $t \to h_t$, h_t^i must then vanish on $x_i = 0$ in the neighborhood of $t = 0$).

Since the proof is the same for all $i = 1, \ldots, r$, we consider $i = 1$ only.

We know that $\frac{d}{dt} h_t^1(x) = \xi_1(h_t(x))$, $\frac{d}{dt} h_t^1|_{t=0} = \xi_1(x) = x_1 \widetilde{\xi_1}(x)$, and hence $\frac{d}{dt} h_t^1|_{t=0}$ vanished at $x = 0$. Now consider the second derivative:

$$\frac{d^2}{dt^2} h_t^1 = \sum_{j=1}^{n} \frac{\partial \xi_1}{\partial x_j} \frac{dh_t^1}{dt} = \sum_{j=1}^{n} \frac{\partial \xi_1}{\partial x_j} \xi_j(h_t(x))$$

$$= \left(\frac{\partial}{\partial x_1} \xi_1(h_t(x)\right) + x_1 \left(\sum_{i=2}^{n} \frac{\partial \widetilde{\xi_1}}{\partial x_i} \xi_i(h_t(x) \right).$$

Hence $\frac{d^2 h_t^1}{dt^2}|_{t=0} = 0$ at $x_1 = 0$. By induction we conclude that the derivatives of all order vanish at $t = 0$ for $x_1 = 0$. Because h_t^i are analytic therefore $h_t^i \equiv 0$.

Proof of Theorem 3.2

Before giving our proof of the theorem, we need a lemma.

Let F is an infinitesimally versal deformation of the germ $\underline{f}$ and let ϕ be any one-parameter deformation of F :

$\Phi(x, \lambda, 0) \equiv F(x, \lambda), \quad F(x, 0) \equiv \underline{f}(x)$

$\Phi\colon (\mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}, 0) \to (\mathbb{R}^q, 0)$.

We may consider Φ as an $l+1-$parameter deformation of the germ $\underline{f}$ of a map from $\mathbb{R}^n$ into $\mathbb{R}^q$ with parameters $\lambda \in \mathbb{R}^l,\ u \in \mathbb{R}$.

Lemma (Reduction) *The deformation Φ of $\underline{f}$ is equivalent to one induced from F.*

Proof We construct a vector field germ v at 0 in space $\mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}$. In such a way that

i) $v = \frac{\partial}{\partial u} + \varepsilon(\lambda, u)\frac{\partial}{\partial \lambda} + X(x, \lambda, u)\frac{\partial}{\partial x}$

ii) $v \circ \Phi = 0$.

According to i) the phase curves of such a field are transversal to the hyperplane $u = 0$ and determine near zero a smooth fibration of $n+l+1-$dimensional space over $n+l-$dimensional space. This fibration may be described as follows. Associate to each point (x, λ, u) the intersection of the phase curve through it with plane $u = 0$. Denote the $x-$and $\lambda-$coordinate of this point of intersection by g and φ. According to i) the fibration so constructed can be written in the form $(x, \lambda, u) \mapsto (g(x, \lambda, u), \varphi(\lambda, u))$. By ii) it is clear that Φ is constant deformation of F, hence v is constant of vector field

$\eta = \frac{\partial}{\partial u} + \varepsilon(\lambda, u)\frac{\partial}{\partial \lambda}$ in neighbourhood of $0 \in \mathbb{R}^l \times \mathbb{R}$.

If $\psi(u, t_1, \ldots, t_l) = (u, \psi_u(t_1, \ldots, t_l))$ is locally dif and only if eomorphism defined by in integral of vector field η. By theorem 3.3 we have $\psi_n(t_1, \ldots, t_l)$ is a germ of holomorphic function, then deformation Φ of $\underline{f}$ is $r-$reticularly equivalent to one induced from F for $h\colon (\mathbb{R}^l \times \mathbb{R}, 0) \to (\mathbb{R}^l, 0)\ \ (u, t_1, \ldots, t_l) \mapsto \psi_u^{-1}(t_1, \ldots, t_l)$.

Therefore to prove the lemma, it remains to construct a field v satisfying i) and ii).

Since F is an infinitesimally versal deformation of $\underline{f}$ then $\forall \alpha \in \varepsilon(n+l, p)$ we have

$$\alpha(x) = \sum_{i=1}^{r} x_i \frac{\partial f}{\partial x_i} h_i(x) + \sum_{i=r+1}^{n} \frac{\partial f}{\partial x_i} h_i(x) + \sum_{i=1}^{l} \dot{F}_i\ \xi_i.$$

Consequently for every variation $\alpha(x, \lambda, u)$ of Φ there exists a decomposition:

$$\alpha(x, \lambda, u) = \sum_{i=1}^{r} x_i \frac{\partial f}{\partial x_i} h_i(x) + \sum_{i=r+1}^{n+1} \frac{\partial f}{\partial x_i} h_i(x) + \sum_{i=1}^{l} \dot{F}_i \, \xi_i.$$

The preparation theorem (see section 6.6, p.130 of [2]) shows that the decomposition exists also for convergent series and in the $C^\infty-$case [it is necessary to apply the theorem to the $A_{x,\lambda,u}-$module $(A_{x,\lambda,u})^n / \{\frac{\partial \Phi}{\partial x_i},\ \Phi_i e_j\}$, to the map $(x, \lambda, u) \to (\lambda, u)$ and to generators $\frac{\partial \Phi}{\partial \lambda_i}$].

The decomposition as above for $\alpha = -\frac{\partial \Phi}{\partial u}$ supplies the desired solution, the lemma is proved.

The continuation of the proof of theorem 3.2

Let F' be any deformation of f with parameter $\lambda' \in \mathbb{R}^{l'}$ and let F be an infinitesimally versal deformation of the same germ with parameter $\lambda \in \mathbb{R}^l$. From the "sum" that is the deformation $\hat{F}(x, \lambda, \lambda') \equiv F(x, \lambda) + F'(x, \lambda') - \underline{f}(x)$ with $l + l'-$dimensional parameter (λ, λ').

For $\lambda' = 0$ the deformation $\hat{F}$ reduces to F and for $\lambda = 0$ to F'. The inclusion of submanifold in the base of the deformation induces a deformation whose base is the embedded submanifold; we shall call the original deformation (with the large base) an extension of the deformation with the smaller base. Note that an extension of an infinitesimally versal velocities only increase.

Consider now the chain of subspace $\mathbb{R}^l \subset \mathbb{R}^{l+1} \subset \cdots \subset \mathbb{R}^{l+l'}$, beginning with the base of the deformation F and finishing with the base of the deformation $\hat{F}$. The restrictions of $\hat{F}$ to these subspaces are infinitesimally stable. Consequently applying the reduction lemma we may see that deformation $\hat{F}$ is $r-$reticularly equivalent to one induced from F. But the deformation F' is induced from $\hat{F}$. Therefore the deformation F' also is $r-$reticularly equivalent to one induced from F, at which point therefore F is the versal unfolding the proof of the theorem is concluded.

Now we consider the case when F is versal we see that $\mathrm{Cod}_r(f)$ is finite. Indeed, it follows readily from F is a versal unfolding of $\underline{f}$ that for all $\alpha(x) \in \varepsilon(n, m)$, we construct an 1-parameter unfolding $F'(x, t) = \underline{f}(x) + \alpha(x)$ of $\underline{f}$. Since F is versal hence there exist g, φ such that $\underline{f}(x) + t\alpha(x) = F[g(x, t), \varphi(t)]$, where $g : \underline{X} \times I\!R \to \underline{X}$, $g(x, 0) = x$ and $\varphi : I\!R \to I\!R^l$, $\varphi(0) = 0$ such that $\underline{f}$ and g are smooth germs. Dif and only if erentiating by t at zero we get

$$\alpha(x) = (F[g(x, t), \varphi(t)])'|_{t=0} = \sum_{i=1}^{r} x_i \frac{\partial F}{\partial x_i} \frac{\partial g_i(x, t)}{\partial t}\bigg|_{t=0} +$$

$$\sum_{i=r+1}^{n} \frac{\partial F}{\partial x_i} \frac{\partial g_i(x, t)}{\partial t}\bigg|_{t=0} +$$

$$\sum_{i=1}^{l} \frac{\partial F}{\partial x_i} \left.\frac{\partial \varphi_i}{\partial t}\right|_{t=0} = \sum_{i=1}^{r} x_i \frac{\partial \underline{f}}{\partial x_i} h_i(x) + \sum_{i=r+1}^{n} \frac{\partial \underline{f}}{\partial x_i} h_i(x) + \sum_{i=1}^{l} c_i \; \dot{F}_i \; (x),$$

where

$$h_i(x) = \left.\frac{\partial g_i(x,t)}{\partial t}\right|_{t=0}, \; i = 1, 2, \ldots, n; \;\; \dot{F}_j \; (x) = \frac{\partial F}{\partial u_j}, j = 1, 2, \ldots l;$$

$$c_i = \left.\frac{\partial \varphi_i(x,t)}{\partial t}\right|_{t=0}.$$

Therefore F is an infinitesimally versal l-parameter unfolding of $\underline{f}$, then the equality (1) holds and it implies, of course, the finiteness of $\mathrm{Cod}_r(\overline{f})$.

3.4 Theorem *$\underline{f}$ is infinitesimally stable if and only if $\underline{f}$ is stable.*

Proof Firstly, we note that $\underline{f}$ is infinitesimally stable if and only if $\mathrm{Cod}_r(f) = 0$ (by definition). Now assume that $\underline{f}$ is stable, we have to prove that $\mathrm{Cod}_r(f) = 0$. This mean that for all $g \in \varepsilon(n,m)$ we have $g \in T_r f$. Because $\underline{f}$ is stable, any unfolding of $\underline{f}$ is trivial, in particular the one-parameter unfolding F defined by $F(x,u) = (\underline{f}(x) + ug(x), u)$ is trivial. Therefore there exist $H \in \mathrm{Diff} andonlyif^r(n+1)$, $G : I\!R^{m+1} \to I\!R^{m+1}$, (without loss of generality we can assume that $G = \mathrm{Id}_{I\!R^{m+1}}$) such that $F = G \circ F' \circ H$, where H is of the form

$$H(x,u) = (x_1 h_1(x,u), \ldots, x_r h_r(x,u), h_{r+1}(x,u), \ldots, h_n(x,u), u)$$

and F' is the constant unfolding of $\underline{f}$. Hence we have

$$\underline{f}(x) + ug(x) = \underline{f}[h_1(x,u), h_2(x,u), \ldots, h_n(x,u)]; \tag{5}$$

$$h_i(x,u) = x_i \tilde{h}_i(x,u), \; i = 1, \ldots, r.$$

Differentiating both sides of (5) by u at $u = 0$ we get the equality

$$g(x) = \sum_{i=1}^{r} x_i \frac{\partial \tilde{h}_i}{\partial u} \left.\frac{\partial f}{\partial x_i}\right|_{u=0} + \sum_{i=r+1}^{n} x_2 \frac{\partial h_i}{\partial u} \left.\frac{\partial f}{\partial x_i}\right|_{u=0}.$$

Hence $g \in T_r(f)$.

If $\underline{f}$ is infinitesimally stable, then $\mathrm{Cod}_r(f) = 0$. Now if we consider $\underline{f}$ as an unfolding of $\underline{f}$ with base $u = \{0\}$, then $\underline{f}$ is versal, and so any unfolding F of $\underline{f}$ is derived from $\underline{f}$ by a change of basis. Therefore F is trivial, and we conclude that $\underline{f}$ is stable.

3.5 Theorem *An unfolding F of $\underline{f}$ is versal if and only if F is stable.*

Proof Let F be an unfolding of $\underline{f}$. Then F is stable if and only if $\mathrm{Cod}_r(f) = 0$ (see 3.4), hence it is enough to prove that F is versal if and only if $\mathrm{Cod}_r(F) = 0$.

We use the Weierstrass preparation theorem (in the algebraic version) (see [4]).

Let $F(x, u) = (\widetilde{F}(x, u), u)$, let $g_1, \ldots g_s \in O(n + q, p)$ and put $g_{i,0}(x) = g_i(x, 0)$; of course $g_{i,0} \in O(n, p)$. Then the following conditions are equivalent:

a) $T_r f + \mathbb{R}\{g_{1,0}, \ldots, g_{s,0}\} = O(n, p)$.

b) $O(n + p)\{x_1 \frac{\partial \widetilde{F}}{\partial x_1}, \ldots, x_r \frac{\partial \widetilde{F}}{\partial x_r}, \frac{\partial \widetilde{F}}{\partial x_{r+1}}, \ldots \frac{\partial \widetilde{F}}{\partial x_n}\} +$
$O(n + p)\{e_1, \ldots, e_p\} + O(p)\{g_1, \ldots g_s\} = O(n + q, p)$.

By the notation of $\mathrm{Cod}_r(F)$, we have $\mathrm{Cod}_r(F) = 0$.

Problem We have generalized Mather's theory of singularities of map-germs to reticular map-germs. In [10] D. Siersma classified the germs of function with boundary and with corners in small dimensions only. Hence the interesting problem in singularity theory is to classify the reticular map-germs in higher dimensions?

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