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A SMALLER COVER FOR CONVEX UNIT ARCS

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Abstract

The Moser's worm problem asks for the smallest set on the plane that can cover every unit arc. The smallest cover known is by Norwood and Poole of which the area is 0.260437. An interesting variant of this problem is to find the smallest cover for every convex unit arc. Thirty years ago, Wetzel proved that the isosceles right triangle with unit hypotenuse and area 0.25 is such a cover. Recently, Johnson and Poole found a convex cover of area 0.2466. In this work, we establish a smaller cover of area 0.2464 obtained from clipping the triangle at height 0.44.

1 Introduction

In 1966, Leo Moser set a well-recognized geometry problem on the plane called the ***Moser's worm problem*** [7]. The problem is to find the smallest set that can cover every unit arc. Naturally we measure a set by its area. The disk of radius $\frac{1}{2}$ is clearly capable of covering all unit arcs. It has area 0.78540. In 1971, Wetzel published the proof by Meir that a semidisk of radius $\frac{1}{2}$, whose area is 0.39270, is also such a cover and added a family of sectorial covers of which the smallest one has area 0.34501 [13]. He also conjectured that the 30-degree sector of unit radius (area 0.26180) is a cover. Many questions arised concerning necessary conditions of covers for unit arcs. One is to find good lower bounds of the width of these covers. An answer came from the discovery

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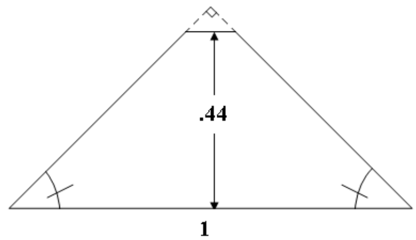

Figure 2.1: A smaller cover for convex unit arcs of area 0.2464.

of the unit ***broadworm*** [11]. It is the unique unit arc with the maximum width of $b_0 = 0.438925$. This leads to a lower bound of the area of convex covers $b_0/2 = 0.21946$ [12] [1]. Later, Gerriets found a cover of area 0.3214 composing of a semiellipse and a triangle glued together [2]. Later in 1972, Gerriets and Poole showed that a rhombus of unit major axis (area 0.2887) can cover unit arcs [4]. The rhombus can be clipped to get a smaller cover of area 0.28610 [5]. In 1989, Norwood, Poole and Laidacker rounded up a corner of the rhombus to get a smaller cover of area 0.27524 [10]. Recently, in 2002, Norwood and Poole established the current record with a nonconvex cover of area 0.260437 and also provided a smaller convex cover of area 0.2738086 [9]. One challenging problem is to prove that the sector conjectured by Wetzel can cover all unit arcs. Currently, it is the smallest convex contender.

One interesting variant of the Moser's worm problem is to find the smallest set that can cover all convex unit arcs. In 1970's, Wetzel noted that an isosceles right triangle T with unit hypotenuse (area 0.25) can cover every convex unit arc [6]. He showed that a convex unit arc can be placed inside the triangle T in its "natural standing orientation" as shown in Figure 3.1. In addition, he showed that a smaller cover can be obtained from clipping the right angle of T to give a cover with area approximately 0.2492. At the time he conjectured that the triangle T clipped at the minimum width of b_0 is also a cover. In 2002, Johnson, Poole and Wetzel found a sophisticated way to prune T near its right angle by 2 symmetric parabolas to form a cover with area about 0.2466 [6]. Wetzel proved that many other arcs can also be covered by these covers for convex arcs [8].

Our proof. Our proof is based on simple comparisons using translations and then numerical minimizations by Mathematica (version 5 or later). The minimizations are called convex programming as we minimize convex functions on convex domains of which the convergences are confirmed theoretically.

2 The cover

Let $\overline{T}$ be the clipped triangle T in Figure 2.1 where its height is $h_0 = 0.44$. The top edge has length $1 - 2h_0$ and each side edge has length $\sqrt{2}h_0$. It has area 0.2464. We will show later that it can cover every convex unit arc.

3 The proof

In the proof of the main theorem, we will use the following 3 lemmas. We state the first lemma without proof.

Lemma 3.1. *On the plane, let l be a line and P and Q be 2 distinct points on the same side of l. Then there is only one point R on l that minimizes the sum of the distances $PR+RQ$. Furthermore, R is the only point for which the segments $\overline{PR}$ and $\overline{RQ}$ make the same angle with l.*

This lemma is also true when P and Q are not on the same side of l. When P and Q are on l, the set of all minimum points are the segment PQ. When P and Q are on opposite sides of l or when either P or Q is on l, the minimum point is the intersection of l and PQ.

At each point on a closed convex arc, there is a tangent line that the arc lies on only one of its sides. In many cases, there are many choices of such lines. When a convex arc is not closed, it has 2 endpoints. Each convex arc is on one side of the line L through its endpoints. Consider the 2 lines through the 2 endpoints perpendicular to L. These 2 lines are parallel. A ***drape*** [6] is a convex arc that stays in between the 2 perpendicular lines. The next lemma tells us that if a convex set can cover every unit drape, then it can cover every convex unit arc.

Lemma 3.2. *[6] A convex unit arc can be covered by the convex hull of a unit drape.*

In 1970's, Wetzel proved that

Lemma 3.3. *[6] The triangle T can cover every convex unit arc in its standing orientation as shown in Figure 3.2.*

Now we are ready to prove the main theorem.

Theorem 3.4. *The clipped triangle $\overline{T}$ is a cover for every convex unit arc.*

Proof By Lemma 3.2, it suffices to show that $\overline{T}$ can cover every unit drape. Suppose for a contradiction that γ is a unit drape that cannot be covered by $\overline{T}$. We will show that its length is greater than one. Throughout the proof, we fix the orientation of γ in its standing orientation as illustrated by Figure 3.1. Let h be the height of γ. We will rotate $\overline{T}$ to cover γ. To describe the orientation

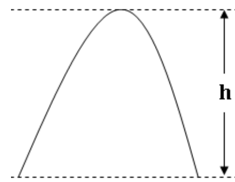

Figure 3.1: The height h of a standing convex arc.

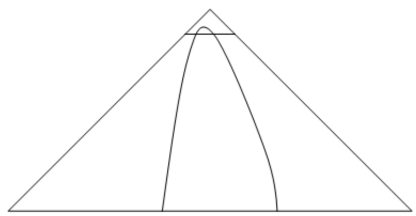

Figure 3.2: The drape γ can be covered by T but cannot be covered by $\overline{T}$.

of $\overline{T}$, we define $\overline{T}_\theta$ to be an isometric copy of $\overline{T}$ after being rotated by angle θ. Furthermore, by "$\overline{T}_\theta$ can cover γ" we mean that $\overline{T}_\theta$ contains a translated copy of γ.

Since T can cover γ in its standing orientation but $\overline{T}$ cannot cover γ, we must have $h > h_0$ (see Figure 3.2). Now we name 5 key points of γ as its head, shoulders and feet as follows. Consider Figure 3.3.

Put a support angle of $\frac{\pi}{4}$ on γ. Let L_s be a point where γ touches the left support line and define R_s similarly. Note that in some case there are more than one choice of L_s. Let H, L_f and R_f be the top, the left and the right ends of γ. Note that we may have $H = L_s, H = R_s, L_s = L_f$ or $R_s = R_f$, but it is clear that $H \neq L_f$ and $H \neq R_f$.

First we place γ in $\overline{T}_{\frac{11}{8}\pi}$ as illustrated in Figure 3.4. Next we see that $R_f \notin \overline{T}_{\frac{11}{8}\pi}$. For if this is not the case, then $L_f \notin \overline{T}_{\frac{11}{8}\pi}$, which implies that L_f is lower than R_f, a contradiction. It follows that $L_f \notin \overline{T}_{\frac{5}{8}\pi}$. Let R be the rhombus with side length $\sqrt{2}h_0$ and angle $\frac{\pi}{4}$ and place γ in R as shown in Figure 3.5. Hence R has width $w = 2\sqrt{2}h_0 \sin\frac{\pi}{8}$ (from left to right.) We will divide into cases according to whether $L_f, R_f \in R$. We will show that the length of γ is greater than 1 in every case.

CASE $L_f, R_f \notin R$ and $L_f R_f \geq w$.

According to a simple direct comparison using 2 translations as in Figure 3.6, we have $1 \geq L_f H + H R_f \geq 2\sqrt{h_0^2 + (\frac{w}{2})^2} = 2h_0\sqrt{1 + 2\sin^2\frac{\pi}{8}} > 1.0006$, a contradiction.

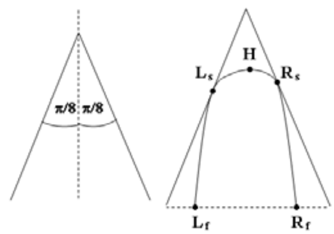

Figure 3.3: After putting the drape γ in a corner of angle $\frac{\pi}{4}$, we name 5 key points.

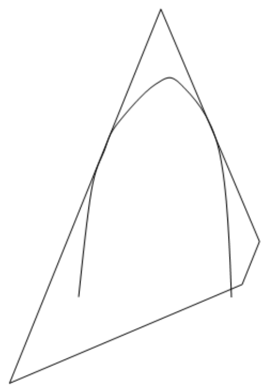

Figure 3.4: The drape γ is placed in $\overline{T}_{\frac{11}{8}\pi}$.

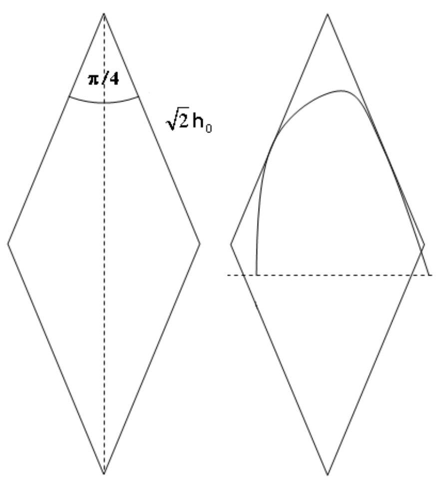

Figure 3.5: The rhombus R with side length $\sqrt{2}h_0$ and angle $\frac{\pi}{4}$.

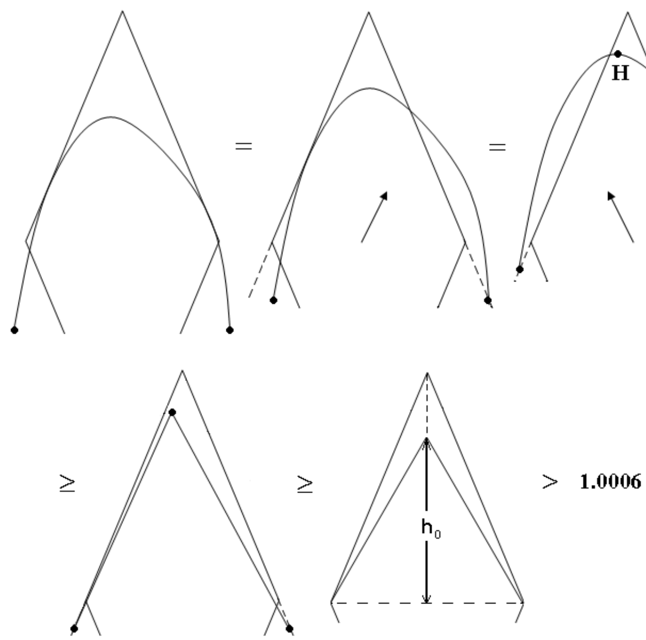

Figure 3.6: Comparison of length of the arcs when $L_f, R_f \notin R$ and $L_f R_f \geq w$.

CASE $L_f, R_f \notin R$ and $L_f R_f < w$.

According to a simple direct comparison using 2 translations as in Figure 3.7, we have $1 \geq \min\{d(x_{l1}, y_{l1}, x_{l2}, y_{l2}, x_H, y_H, x_{r2}, y_{r2}, x_{r1}, y_{r1}) \mid y_{l1} = y_{l2}, (x_{l1}, y_{l1}) \in l_1, (x_{l2}, y_{l2}) \in l_2, y_H \geq y_{l1} + h_0, (x_{r2}, y_{r2}) \in r_2$ and $(x_{r1}, y_{r1}) \in r_1\} = \min\{2d(x_{l1}, y_{l1}, x_{l2}, y_{l2}, 0, y_{l1} + h_0) \mid (x_{l1}, y_{l1}) \in l_1, (x_{l2}, y_{l2}) \in l_2\}$ where $d(x_1, y_1, \ldots, x_n, y_n) = d(x_1, y_1, \ldots, x_{n-1}, y_{n-1}) + d(x_{n-1}, y_{n-1}, x_n, y_n)$, $d(x_1, y_1, x_2, y_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ and l_1, l_2, r_1, r_2 are sides on R. The length of the last 4-segment arc is convex on free variables x_{l2} and y_{l1} that form convex domain. Hence numerical minimization by Mathematica can get closely enough to the minimum. The length of the last 4-segment arc is clearly smooth on y_{l1} as illustrated in Figure 3.8, where the variable $y_{l1} \in [-\sqrt{2}h_0 \sin \frac{7}{8}\pi, 0]$, and the minimum is 1.00004, a contradiction.

CASE $R_f \in R$ (or $L_f \in R$).

By the previous case, we have $R_f \notin \overline{T}_{\frac{11}{8}\pi}$. According to a simple direct comparison as in Figure 3.9, we have $1 \geq \min\{y_{l2} - y + d(x_{l2}, y_{l2}, x_H, y_H, x_{r2}, y_{r2}) + y_{r2} - y \mid (x_{l2}, y_{l2}) \in l_2, y_H \geq y + h_0$ and $(x_{r2}, y_{r2}) \in r_2\} = \min\{2(y_{l2} - y + d(x_{l2}, y_{l2}, 0, y + h_0)) \mid (x_{l2}, y_{l2}) \in l_2\}$. The length of the last 4-segment arc is convex on free variables x_{l2} and y that form convex domain. Hence numerical minimization by Mathematica can get closely enough to the minimum. The length of the last 4-segment arc is clearly smooth on y as illustrated in Fig-

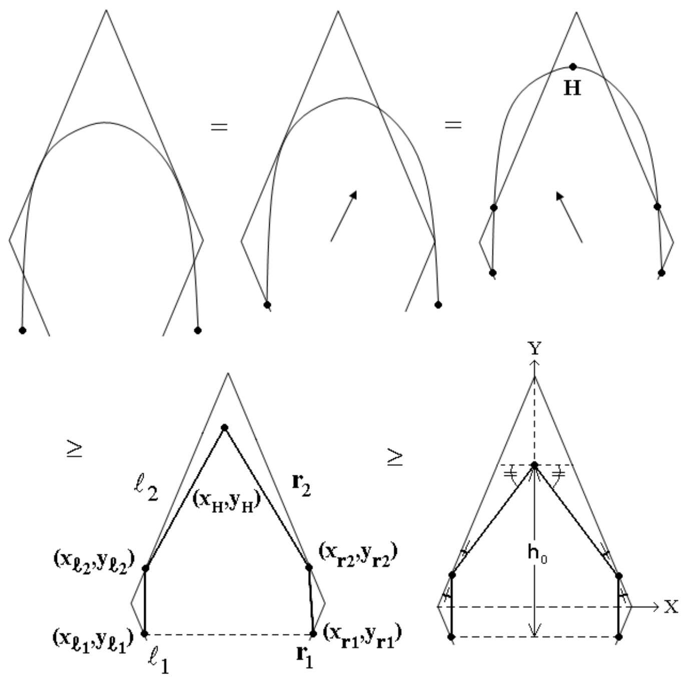

Figure 3.7: Comparison of length of the arcs when $L_f, R_f \notin R$ and $L_f R_f < w$.

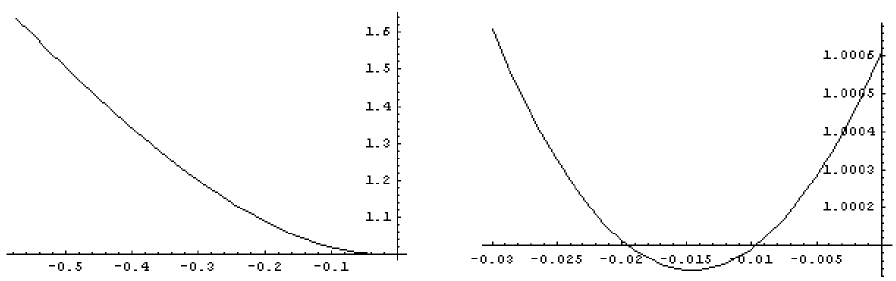

Figure 3.8: The graph of the minimum length of the last arc in the previous figure.

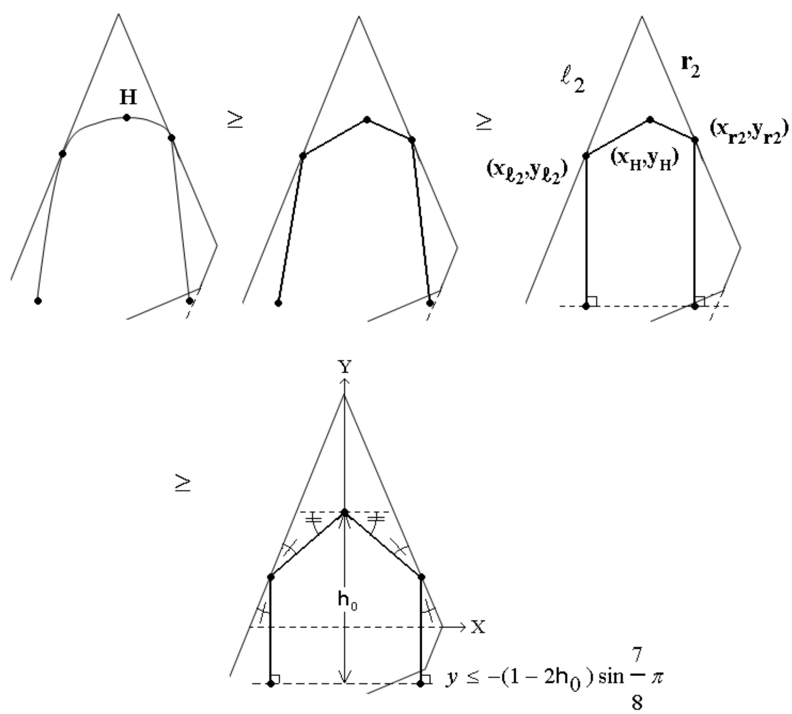

Figure 3.9: Comparison of length of the arcs when $R_f \in R$.

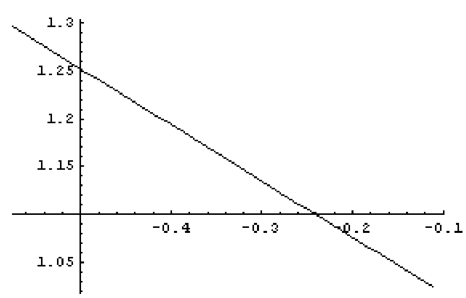

Figure 3.10: The graph of the minimum length of the last arc in the previous figure.

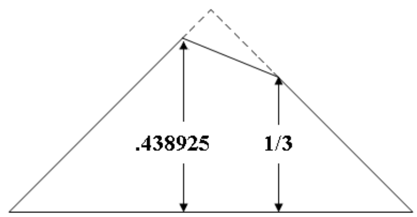

Figure 3.11: A clipped triangle that might cover every convex unit arc.

ure 3.10, where the variable $y \in [-\sqrt{2}h_0 \sin \frac{7}{8}\pi, -(1-2h_0)\sin \frac{7}{8}\pi]$, and the minimum is 1.02396, a contradiction.

In any case, we found a contradiction. Therefore γ can be covered by $\overline{T}$. $\square$

We can see that h_0 can be lowered a little bit as long as the minimum length in the second case, currently 1.00004, is still greater than 1.

We conjecture that the clipped triangle in Figure 3.11 can cover every convex unit arc. It has no symmetry and has area 0.23982 which is close to the lower bound 0.21946. A similar cover was conjectured earlier in [6]; see Figure 9(b).

4 Mathematica code and output

In Figure 4.1, it shows the mathematica code for calculations of the 3 cases. It runs in Mathematica version 5 or later as it heavily uses the command "Minimize".

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```
In[1]:= h0 = .44; θ = π/8;
        s = Sin[θ]; c = Cos[θ]; tan = Tan[θ]; cot = Cot[θ]; w2 = √2 h0 s;
        xl1 = -tan yl1 - w2; pl1 = {xl1, yl1};
        yl2 = cot (xl2 + w2); pl2 = {xl2, yl2};
        d[p1_, p2_] := √((p1[[1]] - p2[[1]])^2 + (p1[[2]] - p2[[2]])^2);
        d2[p1_, p2_, p3_] := d[p1, p2] + d[p2, p3];

In[7]:= (*********** calculation 1 ****************)
        2 √(h0^2 + w2^2)

Out[7]= 1.00061

In[8]:= (*********** calculation 2 ****************)
        Plot[Minimize[2 d2[pl1, pl2, {0, yl1 + h0}], xl2][[1]], {yl1, -√2 h0 c, 0}];
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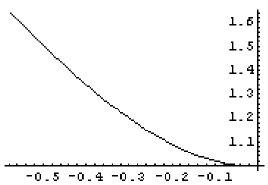


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In[9]:= Plot[Minimize[2 d2[pl1, pl2, {0, yl1 + h0}], xl2][[1]], {yl1, -.03, 0}];
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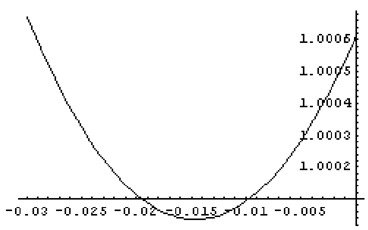


```
In[10]:= Minimize[2 d2[pl1, pl2, {0, yl1 + h0}], {yl1, xl2}]

Out[10]= {1.00004, {xl2 → -0.211418, yl1 → -0.0146073}}

In[11]:= (*********** calculation 3 ****************)
         -(1 - 2 h0) c

Out[11]= -0.110866

In[12]:= Plot[Minimize[2 (Abs[yl2 - y] + d[pl2, {0, y + h0}]), {xl2}][[1]], {y, -√2 h0 c, -(1 - 2 h0) c}];
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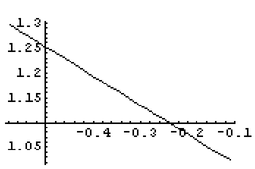


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In[13]:= Minimize[2 (Abs[yl2 - y] + d[pl2, {0, y + h0}]), y < -(1 - 2 h0) c, {y, xl2}]

Out[13]= {1.02396, {xl2 → -0.173774, y → -0.110866}}
```

Figure 4.1: Mathematica code for calculations of the 3 cases.

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