A $3\text{-}(v, 4, 1)$ PACKING CONSTRUCTION

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Abstract

The packing number $D(v, k, t)$ is the maximum number of blocks for any $t\text{-}(v, k, 1)$ packing. A construction for $3\text{-}(v, 4, 1)$ is given, which when applied to $v \equiv 5(\mathrm{mod}\ 6)$ produces packings which can improve the packing number for $2v + 1$ in certain cases. In some cases, it can achieve the bound e.g. it is shown that $D(27, 4, 3) = 702$.

1 Introduction

Let $v \geq k \geq t$. A $t\text{-}(v, k, \lambda)$ *packing* is a pair $(X, \mathcal{B})$, where X is a v-set of elements (*points*) and $\mathcal{B}$ is a collection of k-subsets of X (*blocks*), such that every t-subset of points occurs in at most λ blocks in $\mathcal{B}$. The *packing number* $D_\lambda(v, k, t)$ is the maximum number of blocks in any $t\text{-}(v, k, \lambda)$ packing. A t-(v, k, λ) packing $(X, \mathcal{B})$ is *optimal* if $|\mathcal{B}| = D_\lambda(v, k, t)$. If $\lambda = 1$, we usually write $D(v, k, t)$ for $D_1(v, k, t)$.

Upper bounds for $D_\lambda(v, k, t)$ were given by Johnson cited from [1; page 409-410] as stated in the following theorems.

Theorem 1.1. $D_\lambda(v, k, t) \leq \lfloor \frac{v \cdot D_\lambda(v-1, k-1, t-1)}{k} \rfloor$. *Iterating this bound yields* $D_\lambda(v, k, t) \leq U_\lambda(v, k, t)$, *where* $U_\lambda(v, k, t) = \lfloor \frac{v}{k} \lfloor \frac{v-1}{k-1} \cdots \lfloor \frac{\lambda(v-t+1)}{k+t-1} \rfloor \rfloor \rfloor$.

Further, if $\lambda(v - 1) \equiv 0(\mathrm{mod}\ (k - 1))$ and $\lambda v(v - 1) \equiv -1(\mathrm{mod}\ k)$, then $D_\lambda(v, k, 2) \leq U_\lambda(v, k, 2) - 1$.

Theorem 1.2. *Suppose $d = D_\lambda(v, k, t) = qv + r$, where $0 \leq r \leq v - 1$. Then $q(q - 1)v + 2qr \leq (t - 1)d(d - 1)$. From this it follows that $D_\lambda(v, k, t) \leq \lfloor \frac{v(k+1-t)}{k^2 - v(t-1)} \rfloor$.*

This note is motivated by a remark in Stinson ([1]; page 410) that for $v \equiv 5(\mathrm{mod}\ 6)$, $v \geq 17$, essentially nothing is known about the number $D(v, 4, 3)$, the packing number for a 3-$(v, 4, 1)$ packing and by the well known construction of SQS$(2v)$ from SQS(v), see for example Lindner and Rodger [2].

For the most part, our notation and terminology follows that of Lindner and Rodger [2].

2   Construction from v to $2v$

Observe that the well known $2v$ construction for SQS to produce SQS$(2v)$ from an SQS(v), can be applied for packing and for any even integer v where a packing exists.

Theorem 2.1. *A 3-$(v, 4, 1)$ packing $(X, \mathcal{B})$ for even v implies the existence of a 3-$(2v, 4, 1)$ packing.*

Proof   A 3-$(v, 4, 1)$ packing $(X, \mathcal{B})$ is given. Let $\mathcal{F} = \{F_1, F_2, \cdots, F_{v-1}\}$ be a 1-factorization of K_v defined on X. Then the following blocks together produce a 3-$(2v, 4, 1)$ packing on the set of vertices $X \times \{1, 2\}$ and set of blocks $\mathcal{B}'$, where $\mathcal{B}'$ is defined as follows.

1. For each block $B \in \mathcal{B}$, $B \times \{1\}, B \times \{2\} \in \mathcal{B}'$.

2. For each $F_j \in \mathcal{F}$, $\{(a, 1), (b, 1), (c, 2), (d, 2)\} \in \mathcal{B}'$ if and only if $\{a, b\} \in F_j$ and $\{c, d\} \in F_j$.

$\square$

Corollary 2.2. *The construction produces optimal packing for $v \equiv 0(\mathrm{mod}\ 6)$*

Proof   Let $v = 6t$. Then $D(6t, 4, 3) = \lfloor \frac{18t^3 - 9t^2 - 3t}{2} \rfloor = \frac{18t^3 - 9t^2 - 3t}{2}$. On the other hand the construction produces $\mathcal{B}'$ with $|\mathcal{B}'| = 2 \cdot \frac{18t^3 - 9t^2 - 3t}{2} + (3t) \cdot (3t) \cdot (6t - 1) = 72t^3 - 18t^2 - 3t$, the exact number needed for optimal packing for 3-$(v, 4, 1)$, i.e, $D(12t, 4, 3)$ is achieved. This means, the construction produces an optimal 3-$(2v, 4, 1)$ packing, just like in the case of SQS(v). $\square$

For example, when $v = 6$, the packing number $D(6, 4, 3)$ is 3. Let $\mathcal{F} = \{F_1, F_2, \cdots, F_5\}$ be a set of 1-factorization of K_6. For each $F_j \in \mathcal{F}$, F_i contains 3 independent edges. Therefore we get a packing for $2v = 12$ with $3 + 3 + 3 \cdot 3 \cdot 5 = 51$ blocks. Note that $D(12, 4, 3)$ is 51.

3   Construction from v to $2v + 1$

A slight modification of above construction is very useful as it can be used to get 3-$(v, 4, 1)$ packing for $v \equiv 5 \pmod 6$.

Theorem 3.1. *Packings* 3-$(v, 4, 1)$ *and* 3-$(v + 1, 4, 1)$ *implies the existence of a* 3-$(2v + 1, 4, 1)$ *packing.*

Proof   Let v be an odd integer. Suppose a packing 3-$(v, 4, 1)$ and a packing 3-$(v + 1, 4, 1)$ with sets of blocks $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, are given. Let $\mathcal{F} = \{F_1, F_2, \cdots, F_v\}$ be a 1-factorization of K_{v+1}. Then the following blocks together produce a 3-$(2v + 1, 4, 1)$ packing on the set of vertices $\{(i, 1) : i = 1, 2, \cdots, v\} \cup \{(i, 2) : i = 1, 2, \cdots, v + 1\}$ with set of blocks $\mathcal{B}'$ according to the following rules.

1. For each block $B \in \mathcal{B}_1$, $B \times \{1\} \in \mathcal{B}'$.

2. For each block $B \in \mathcal{B}_2$, $B \times \{2\} \in \mathcal{B}'$.

3. For each $F_j \in \mathcal{F}$, $\{(a, 1), (b, 1), (c, 2), (d, 2)\} \in \mathcal{B}'$ if and only if $\{a, b\} \in F_j$ and $\{c, d\} \in F_j$.

$\square$

The construction produces $|\mathcal{B}_1| + |\mathcal{B}_2| + \frac{v-1}{2} \cdot \frac{v+1}{2} \cdot v$ blocks. Moreover the construction gives blocks with more structure than the packing obtained from deleting an element from a 3-$(2v + 2, 4, 1)$ and it can produce optimal packing. For example, when $v = 7$, we get $7 + 14 + 3 \cdot 4 \cdot 7 = 105$ blocks, which gives an optimal packing as $D(15, 4, 1) = 105$.

Note that if $v \equiv 5 \pmod 6$, then $2v + 1 \equiv 5 \pmod 6$. Let us first consider in the case when $v = 5$. It is known that $D(5, 4, 3) = 1$ and $D(6, 4, 3) = 3$. Thus by the above construction, we get $D(11, 4, 3) \geq 1 + 3 + 2 \cdot 3 \cdot 5 = 34$. It is also known that $D(11, 4, 3) = 35$. So, the construction achieves one block less than an optimal packing in this case. Now consider when $v = 11$. It is known that $D(11, 4, 3) = 35$ and $D(12, 4, 3) = 51$. Thus by the above construction, we get $D(23, 4, 3) \geq 35 + 51 + 5 \cdot 6 \cdot 11 = 416$. It is known that $D(23, 4, 3) \leq \lfloor \frac{23}{4} \lfloor \frac{22}{3} \lfloor \frac{21}{2} \rfloor \rfloor \rfloor = 419$. Thus $416 \leq D(23, 4, 3) \leq 419$. For $v = 17$, we know that $D(35, 4, 3) \leq 1583$ and by the construction gives that $D(35, 4, 3) \geq 1579$. Finally, it is known that $D(27, 4, 3) = 702$, and the construction of 3-$(27, 4, 1)$ above produces $65 + 91 + 6 \cdot 7 \cdot 13 = 702$ blocks. In fact, we have the following corollary.

Corollary 3.2. *If* $v \equiv 1 \pmod{12}$, *say* $v = 12s + 1$, $D(24s + 3, 4, 3) = (24s + 2)(24s + 3)s = 576s^3 + 120s^2 + 6s$ *and the construction also produces the same number of blocks required to give optimal packing.*

4 Generalization of v to $2v$ construction from SQS(v) to $(2t-1)$-$(v, 2t, 1)$ packing for any t

Suppose large set exists for (v, t), in other words assume that all t-subsets of v-set can be partitioned to BIBD$(v, t, 1)$. Note that the set of all $2t$-subsets of a v-set is a $(v, \binom{v}{t}, \binom{v}{t-1}, t, \binom{v}{t-1} \cdot (\frac{t-1}{v-1}))$. So there are $s = \binom{v}{t-1} \cdot (\frac{t-1}{v-1})$ $BIBD(v, t, (\lambda = 1)$-designs. Call them $F_1, F_2, \cdots, F_s$. We have the following construction.

Theorem 4.1. *Suppose there exists a $(2t-1)$-$(v, 2t, 1)$ packing with M blocks, then there exists a $(2t-1)$-$(2v, 2t, 1)$ packing with $2M + [\frac{t-1}{v-1} \cdot \binom{v}{t-1}]^2 \cdot [\frac{\binom{v}{t}}{\binom{v}{t-1} \cdot \frac{t-1}{v-1}}]$.*

Proof   As usual let $V = \{1, 2, \cdots, v\}$ and the blocks of $(2t-1)$-$(v, 2t, 1)$ packing be $\mathcal{B}$. The blocks of $(2t-1)$-$(2v, 2t, 1)$ are in $\mathcal{B}'$ on the set $V \times \{1, 2\}$, according to the following rules.

1. If $B \in \mathcal{B}$, then $B \times \{1\}$ and $B \times \{2\}$ are in $\mathcal{B}'$.

2. For each F_j, if $\{a_1, a_2, \cdots, a_t\}$ and $\{b_1, b_2, \cdots, b_t\}$ are any two not necessary distinct blocks in F_j, then $\{(a_1, 1), (a_2, 1), \cdots, (a_t, 1), (b_1, 2), (b_2, 2), \cdots, (b_t, 2)\}$ is in $\mathcal{B}'$.

Note that two blocks of size t in F_j intersect in only 1 point. Therefore the intersection between any two distinct blocks directly obtained from $\mathcal{B}$ and from F_j is at most t and the intersection between the blocks from $\mathcal{B}$ can be $(2t-1)$. Intersection of blocks obtained from the same F_j can be at most $(t+1)$. Intersection of blocks obtained from different F_j can be at most $2(t-1) = 2t-2$. $\qquad \square$

For example, let $v = 7$, we have $D(7, 6, 1) = 1$. Since $\binom{7}{3} = 35$, there is a partition of all subsets of size 3 into 5 2-$(7, 3, 1)$. So there exists 5-$(14, 6, 1)$ packing with $1 + 1 + 5 \times 7 \times 7 = 247$ blocks. The bound for 5-$(14, 6, 1)$ is $\lfloor \frac{14}{6} \lfloor \frac{13}{5} \lfloor \frac{12}{4} \lfloor \frac{11}{3} \lfloor \frac{10}{2} \rfloor \rfloor \rfloor \rfloor \rfloor = 326$. The construction does not provide close enough number of blocks but notice that block intersections are much lower than $2t-2$ for many pairs.

References

[1] D. Stinson, *Packings*, The CRC Handbook of Combinatorial Designs, editors C. J. Colbourn and J. H. Dinitz, CRC Press, Boca Raton, Fl. 1996, 409-413.

[2] C. C. Lindner and C. A. Rodger, Design Theory, CRC Press, 1997.