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FRACTIONAL ORNSTEIN-UHLENBECK SIGNAL PROCESSING

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Abstract

In this paper we consider a problem of signal processing where the signal is expressed by a fractional Ornstein-Uhlenbeck process in general form. An explicit form of the signal is derived from a fractional Langevin equation. A method of L^2-approximation is used to find the approximate estimate for the state of the fractional signal and the convergence to the optimal estimate is established.

1 Introduction

It is known that the Ornstein-Uhlenbeck plays a crucial role in telecommunication as an only stationary Gaussian Markov signal with white noise. But a Gaussian non-Markovian signal is also important in some context where the signal leaves a long time influence upon its behavior. A good candidate for expressing this signal property is a fractional Brownian noise. In this paper we consider a problem of signal processing where the signal is a fractional Ornstein-Uhlenbeck by introducing an approximation approach.

Key words: Fractional Ornstein-Uhlenbeck signal, L^2-approximation approach, fractional Brownian motion. 2010 MS: 60H, 93E05.
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1.1 Fractional Brownian motion

A fractional Brownian motion of Mandelbrot form is a centerred Gaussian process $(W_t^H, t \geq 0)$ with covariance function $R(s,t)$ given by

$$R(s,t) = E(W_s^H W_t^H) = \frac{1}{2}(s^{2H} + t^{2H} + |t-s|^{2H}), \tag{1.1}$$

where H is a parameter called Hurst index, $0 < H < 1$.
In the case where $H = \frac{1}{2}$, W_t^H becomes a usual standard Brownian motion.
The process W_t^H can be decomposed as

$$W_t^H = C_H(U_t + B_t^H), \tag{1.2}$$

where U_t is a stochastic process with absolutely continuous trajectory and C_H is a constant depending only on H, $B_t^H = \int_0^t (t-s)^\alpha dW_s$ with $\alpha = H - 1/2$.
We know that W_t^H is a process of long memory with $H \neq \frac{1}{2}$. In (1.2) this property focuses at the second term B_t^H and by this reason, B_t^H is called a fractional Brownian motion of Liouville form. In this paper we consider fractional noise associated with B_t^H. The problem is how to get the optimal state estimation for a fractional signal that is a general fractional Ornstein-Uhlenbeck process $(X_t, t \geq 0)$ satisfying the following equation

$$dX_t = (a(t)X_t + b(t))dt + \sigma dB_t^H, \tag{1.3}$$

where $H > 1/2$, from an observation Y_t given by

$$dY_t = h(X_t)dt + dV_t, \tag{1.4}$$

where V_t is a standard Brownian motion independent of B_t^H, $h_t = h(X_t)$ is a process such that

$$E\int_0^t h_s^2 ds < \infty, \text{ for every } t \geq 0.$$

1.2 Approximation approach

The fractional Brownian motion B_t^H is not a semimartingale, so a fractional signal driven by B_t^H as X_t in (1.3) cannot be solved by the traditional Ito calculus.
An L^2-approximation approach has been introduced in [2] where a process $B_t^{H,\epsilon}$ is considered instead of B_t^H:

$$B_t^{H,\epsilon} = \int_0^t (t-s+\epsilon)^\alpha dW_s, \ \alpha = H - \frac{1}{2}. \tag{1.5}$$

A calculation says to us that $B_t^{H,\epsilon}$ is in fact a semimartingale.

$$dB_t^{H,\epsilon} = \alpha\varphi_s^\epsilon dt + \epsilon^\alpha dW_t, \tag{1.6}$$

where

$$\varphi_t^\epsilon = \int_0^t (t-s+\epsilon)^{\alpha-1} dW_s,\ \alpha = H - \frac{1}{2}.$$

And as shown in [2] we have the following fundamental result on L^2-convergence of semimartingales $B_t^{H,\epsilon}$.

Result: *$B_t^{H,\epsilon}$ converges to B_t in $L^2(\Omega)$ when $\epsilon \to 0$ and we have*

$$\sup_{0\le t\le T} \|B_t^{H,\epsilon} - B_t\|_{L^2} \le K(\alpha)e^{1/2+\alpha}, \tag{1.7}$$

where $K(\alpha)$ is a constant depending only on $\alpha = H - 1/2$.

Moreover a new approach to stochastic integration and stochastic differential equations driven by B_t^H is given in [3] (refer also to [3]-[9]).

2 General fractional Ornstein-Uhlenbeck signal

2.1 Approximate signal equation

Consider again the equation

$$dX_t = (a(t)X_t + b(t))dt + \sigma dB_t^H, H > 1/2, \tag{2.1}$$

where $0 \le t \le T$, coefficients $a(t)$ and $b(t)$ are deterministic continuous function on $[0,T]$.

It is a generalization of fractional stochastic Langevin equation studied in [6] and [7], where our L^2-approximation method has been applied to find its solution. As shown in [10] the solution of (2.1) is a L^1-limit of that of an approximate equation. Now we prove that it is also a L^2-limit.

By replacing B_t^H by $B_t^{H,\epsilon}$ we obtain the approximate equation for the signal X_t as follows

$$dX_t^\epsilon = (a(t)X_t^\epsilon + b(t))dt + \sigma dB_t^{H,\epsilon}, \tag{2.2}$$

where $0 \le t \le T,\ H > 1/2$.

2.2 Approximate equation

Equation (2.2) can be rewritten as follows

$$dX_t^\epsilon = (a(t)X_t^\epsilon + b(t) + \alpha\varphi_t^\epsilon)dt + \sigma\epsilon^\alpha dW_t. \tag{2.3}$$

A method of equation splitting introduced by us in [6, 7] can be applied to (2.3). We can write

$$X_t^\epsilon = X_1^\epsilon(t) + X_2^\epsilon(t), 0 \le t \le T, \tag{2.4}$$

where

$$dX_1^\epsilon(t) = a(t)X_1^\epsilon(t)dt + \sigma\epsilon^\alpha dW_t \tag{2.5}$$

and

$$dX_2^\epsilon(t) = (a(t)X_2^\epsilon(t) + b(t) + \alpha\varphi_t^\epsilon)dt. \tag{2.6}$$

Equation (2.5) is a simple stochastic linear equation of Langevin type and its solution is

$$X_1^\epsilon(t) = e^{\int_0^t a(u)du}(X_1^\epsilon(0) + \sigma\epsilon^\alpha \int_0^t e^{-\int_0^s a(u)du} dW_s). \tag{2.7}$$

And the equation (2.6) is an ordinary differential equation for every fixed ω and its solution is

$$X_2^\epsilon(t) = e^{\int_0^t a(u)du}\big[X_2^\epsilon(0) + \int_0^t b(s)e^{-\int_0^s a(u)du}ds + \sigma\alpha \int_0^t \varphi_s^\epsilon e^{-\int_0^s a(u)du}ds\big]. \tag{2.8}$$

Now combining (2.4), (2.7) and (2.8) and noticing that $\alpha\varphi_s^\epsilon ds + \epsilon^\alpha dW_s = dB_s^{H,\epsilon}$ we can write the approximate signal X_t^ϵ in the form

$$\begin{aligned} X_t^\epsilon &= X_1^\epsilon(t) + X_2^\epsilon(t) \\ &= e^{\int_0^t a(u)du}\big[X_0 + \int_0^t b(s)e^{-\int_0^s a(u)du}ds + \sigma \int_0^t e^{-\int_0^s a(u)du}dB_s^{H,\epsilon}\big], \end{aligned} \tag{2.9}$$

where X_0 is assumed a random variable such that $E|X_0|^2 < \infty$.

3 Convergence to the exact solution

We can see that the equation (2.1) satisfies all conditions of Theorem of existence and uniqueness for solution of a fractional stochastic differential equation given in [3]. We will prove that the approximate signal X_t^ϵ converges to the fractional X_t that is the exact solution of (2.1). Consider two equations

$$dX_t = (a(t)X_t + b(t))dt + \sigma dB_t^H,$$

$$dX_t^\epsilon = (a(t)X_t^\epsilon + b(t))dt + \sigma dB_t^{H,\epsilon}.$$

3.1 Theorem 3.1

X_t^ϵ converges to X_t in $L^2(\Omega)$ uniformly with respect to $t \in [0, T]$.

Proof. We have

$$X_t - X_t^\epsilon = a(t)\int_0^t (X_s - X_s^\epsilon)ds + \sigma(B_t^H - B_t^{H,\epsilon}).$$

Then

$$\|X_t - X_t^\epsilon\| \le M\|\int_0^t (X_s - X_s^\epsilon)ds\| + \sigma\|B_t - B_t^{H,\epsilon}\|, \quad (3.1)$$

where $\|.\|$ denote for L^2-norm and $|a(t)| \le M$ for $t \in [0,T]$, $M > 0$ due to the fact that $a(t)$ is a continuous function.
In account of (1.7) we can see from (3.1) that

$$\|X_t - X_t^\epsilon\| \le M\int_0^t \|X_s - X_s^\epsilon\|ds + \sigma K(\alpha)\epsilon^{\frac{1}{2}+\alpha}, 0 \le t \le T. \quad (3.2)$$

Applying the Gronwall's lemma to (3.2) we get

$$\|X_t - X_t^\epsilon\| \le \sigma K(\alpha)\epsilon^{\frac{1}{2}+\alpha}e^{-at} \quad (3.3)$$

and then

$$\sup_{0\le t\le T}\|X_t - X_t^\epsilon\| \le \sigma K(\alpha)\epsilon^{\frac{1}{2}+\alpha}e^{-aT} \text{ for } a > 0$$

$$\sup_{0\le t\le T}\|X_t - X_t^\epsilon\| \le \sigma K(\alpha)\epsilon^{\frac{1}{2}+\alpha} \text{ for } a < 0$$

So $X_t^\epsilon \longrightarrow X_t$ in $L^2(\Omega)$ uniformly with respect to $t \in [0,T]$. □

3.2 Collorary 3.1

It follows from Theorem 3.1 and the formula (2.9) that the exact signal X_t can be explicitly expressed as

$$X_t = e^{\int_0^t a(u)du}\big(X_0 + \int_0^t b(s)e^{-\int_0^s a(u)du}ds + \sigma\int_0^t e^{-\int_0^s a(u)du}dB_s^H\big). \quad (3.4)$$

4 Best state estimate for signal X_t

4.1 Approximation for best state estimate

Consider now an approximate model for state estimate of the signal X_t^ϵ form the observation Y_t: Signal X_t^ϵ:

$$dX_t^\epsilon = (a(t)X_t^\epsilon + b(t))dt + \sigma dB_t^{H,\epsilon}. \quad (4.1)$$

Observation Y_t:

$$dY_t = h(X_t^\epsilon)dt + dV_t. \quad (4.2)$$

The model (4.1) - (4.2) can be rewritten as follows

$$dX_t^\epsilon = \big(a(t)X_t^\epsilon + b(t) + \alpha\varphi_t^\epsilon\big)dt + \sigma\epsilon^\alpha dW_t, \tag{4.3}$$

$$Y_t^\epsilon = \int_0^t h(X_s^\epsilon)ds + V_t. \tag{4.4}$$

where W_t and V_t are two independent standard Brownian motion.
Let $\mathcal{F}_t^Y$ be the observation $\sigma-$algebra, that is the algebra generated by all random variables Y_s for $s \leq t$:

$$\mathcal{F}_t^Y = \sigma(Y_s, 0 \leq s \leq t).$$

Also, $\mathcal{F}_t^{Y^\epsilon}$ is denoted for the approximate observation $\sigma-$algebra: $\mathcal{F}_t^{Y^\epsilon} = \sigma(Y_s^\epsilon, 0 \leq s \leq t)$ The best state estimation for approximate signal X_t^ϵ denoted by $\widehat{X_t^\epsilon}$ based on observation information given by $\mathcal{F}_t^Y$:

$$\widehat{X_t^\epsilon} = E\big(X_t^\epsilon|\mathcal{F}_t^{Y^\epsilon}\big). \tag{4.5}$$

Denote by ν_t the innovation process that is a $\mathcal{F}_t^{Y^\epsilon}$-martingale:

$$\nu_t = Y_t^\epsilon - \int_0^t \widehat{h_s^\epsilon}ds, \tag{4.6}$$

where $\widehat{h_s} = \widehat{h(X_s)} = E(h(X_s)|\mathcal{F}_s^Y)$, $0 \leq s \leq t$ and by H_t^ϵ the following expression

$$H_t^\epsilon = a(t)X_t^\epsilon + b(t) + \alpha\varphi_t^\epsilon. \tag{4.7}$$

Now we are in position to apply the FKK (Fujisaki-Kallianpur-Kunita) (see [11]) equation to $\widehat{X_t^\epsilon}$ and we have
Theorem 4.1 *The best state estimate $\widehat{X_t^\epsilon}$ is given by the following equation*

$$\widehat{X_t^\epsilon} = \widehat{X_0^\epsilon} + \int_0^t \widehat{X_s^\epsilon H_s^\epsilon}ds + \int_0^t \big[\widehat{X_s^\epsilon h_s} - \widehat{X_s^\epsilon}\widehat{h_s^\epsilon}\big]d\nu_s, \tag{4.8}$$

where the notation $\wedge$ stands for the best state estimate.

4.2 Best state estimation for the exact signal X_t

Now we have to find

$$\widehat{X_t} = E(X_t|\mathcal{F}_t^Y), \tag{4.9}$$

where the signal X_t is given by (3.4).
Consider the best approximate state $\widehat{X_t^\epsilon} = E(X_t^\epsilon|\mathcal{F}_t^{Y^\epsilon})$.
Put $\epsilon = 1/n, n = 1, 2\ldots$ and denote $X^{(n)}$ for X_t^ϵ with $\epsilon = 1/n$.

Then $\widehat{X_t^\epsilon} = \widehat{X_t^{(n)}} = E(X_t^{(n)}|\mathcal{F}_t^{(n)})$ where $\mathcal{F}_t^{Y^\epsilon} = \mathcal{F}_t^{(n)}$ is the $\sigma-$algebra generated by $(X_0, B_s^{(n)}, V_s, s \leq t)$ with

$$B_t^{(n)} = B_t^{H,1/n} = \int_0^t (t-s-1/n)^\alpha dW_s. \qquad (4.10)$$

By a change of variable we have

$$B_t^{(n)} = \int_0^{t-1/n} (t-u)^\alpha dW_u = B_{t-1/n} \text{ and } B_s^{(n)} = B_{s-1/n}. \qquad (4.11)$$

Therefore $\sigma-$algebras $\mathcal{F}_t^{(n)} = \sigma(X_0, B_{s-1/n}, V_s, s \leq t), n = 1, 2, \ldots$ form an inscreasing filtration and $\mathcal{F}_t^{(n)} \nearrow \mathcal{F}_t^Y$.
By applying the elementary inequality

$$|a+b|^2 \leq \frac{1}{2}|a|^2 + \frac{1}{2}|b|^2$$

we can see

$$\begin{aligned} E|E(X^{(n)}|\mathcal{F}_t^{(n)}) - E(X_t|\mathcal{F}_t^Y)|^2 &\leq \frac{1}{2}E|E(X_t^{(n)} - X_t|\mathcal{F}_t^{(n)})|^2 + \\ &\quad \frac{1}{2}E|E(X_t|\mathcal{F}_t^{(n)}) - E(X_t|\mathcal{F}_t^Y)|^2 \\ &\leq \frac{1}{2}E|X_t^{(n)} - X_t|^2 + \frac{1}{2}E|E(X_t|\mathcal{F}_t^{(n)}) - E(X_t|\mathcal{F}_t^Y)|^2. \qquad (4.12) \end{aligned}$$

In the last side of (4.12) we see that when $n \to \infty$ the first term tends to 0 by Theorem 3.1 for $\epsilon = 1/n$ and the second term converges to 0 as well because of a Levy theorem of convergence of conditional expectation.
Finally we can state

Theorem 4.2: *$\widehat{X_t}$ can be considered as $L^2 - \lim$ of $\widehat{X_t^{(n)}}$ when $n \to \infty$*

$$\widehat{X_t} = L^2 - \lim_{n\to\infty} E(X_t^{(n)})|\mathcal{F}_t^{(n)}. \qquad (4.13)$$

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