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MINIMAL REDUCING SUBSPACES OF THE UNILATERAL SHIFT

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Abstract

We introduce a family $\mathcal{T}$ consisting of invertible matrices with exactly one non zero entry in each row and each column. The elements of $\mathcal{T}$ are mutually non commuting, and need not be normal or self adjoint. We consider the operator weighted sequence space $l^2_B(K)$ with a uniformly bounded weight sequence $B = \{B_n\}_{n=0}^{\infty}$ in $\mathcal{T}$. For the unilateral shift S on $l^2_B(K)$, we obtain a complete description of the reducing and minimal reducing subspaces of S.

1 Introduction

Let K be a separable complex Hilbert space, and $l^2(K) = \oplus_0^{\infty} K$ be the orthogonal sum of $\aleph_o$ copies of the Hilbert space K with a scalar product defined by

$$< f, g >= \sum_{n=0}^{\infty} < f_n, g_n >, \;\; f = (f_0, f_1, \dots) \in l^2(K), g = (g_0, g_1, \dots) \in l^2(K).$$

Let $\{e_i\}_{i=0}^{\infty}$ be an orthonormal basis for K. Also for $i, j \in \{0, 1, 2, ...\}$, let $g_{i,j} := (0, ..., e_i, 0, ...)$ where e_i occurs at the j^{th} position. If $\mathbb{N}_0 := \{0, 1, 2, ...\}$ then $\{g_{i,j}\}_{i,j\in\mathbb{N}_0}$ is an orthonormal basis for the Hilbert space $l^2(K)$.
Let $\mathcal{B}(K)$ denote the space of all bounded linear operators on K with norm defined as $\|T\| = \sup_{\|x\|=1} \|Tx\|$ for $T \in \mathcal{B}(K)$. Let us now consider the

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operator weighted sequence space $l^2_B(K)$. To define $l^2_B(K)$, let $B = \{B_n\}_{n=0}^{\infty}$ be a sequence of invertible bounded linear operators on K, and $l^2_B(K) := \{(f_0, f_1, ...) : f_i \in K \text{ and } \sum_{i=0}^{\infty} \|B_i f_i\|^2 < \infty\}$. For $f = (f_i)$ and $g = (g_i)$ in $l^2_B(K)$, we have

$$\langle f, g\rangle_B := \sum_{i=0}^{\infty} \langle B_i f_i, B_i g_i\rangle \text{ and } \|f\|_B^2 = \sum_{i=0}^{\infty} \|B_i f_i\|^2.$$

As $\|g_{i,j}\|_B = \|B_j e_i\|$, so if $f_{i,j} := \frac{g_{i,j}}{\|B_j e_i\|}$, then $\{f_{i,j}\}_{i,j\in\mathbb{N}_0}$ is an orthonormal basis for the Hilbert space $l^2_B(K)$. If $dim\, K = 1$, then each B_n is a non zero scalar β_n, and $l^2_B(K)$ is the scalar weighted sequence space l^2_β defined in [4].

Depending on the weights $\{B_n\}$ we can have different types of operator weighted sequence spaces $l^2_B(K)$. In this paper we consider the weights B_n to belong to a special subset $\mathcal{T}$ of $\mathcal{B}(K)$ defined as follows:
$\mathcal{T} := \{T \in \mathcal{B}(K)\big|$ T is invertible in $\mathcal{B}(K)$ and the matrix of T with respect to $\{e_n\}_0^{\infty}$ has exactly one non zero entry in each row and each column$\}$
We observe the following:
(i) If $T_1, T_2 \in \mathcal{T}$, then $T_1T_2 \in \mathcal{T}$. However, T_1 and T_2 need not commute and hence elements of $\mathcal{T}$ are not simultaneously diagonalizable with respect to $\{e_n\}_0^{\infty}$.
(ii) If $T \in \mathcal{T}$ then its Hilbert adjoint T^* and inverse T^{-1} are also in $\mathcal{T}$.
(iii) Elements of $\mathcal{T}$ may not be self adjoint or normal.

The unilateral shift S on $l^2_B(K)$ is defined as $S(f_0, f_1, ...) = (0, f_0, f_1, ...)$. Clearly, S is bounded if and only if $\sup_{i,j} \frac{\|B_{j+1}e_i\|}{\|B_j e_i\|} < \infty$. Here we will determine the minimal reducing subspaces of S on $l^2_B(K)$, where the weight sequence $B = \{B_n\}$ is in $\mathcal{T}$. We recall that for a bounded linear operator T on a Hilbert space H, a subspace M of H is said to be invariant for T if $T(M) \subseteq M$. If the subspace M is invariant for both T and T^*, then M is said to be reducing for T. A reducing subspace M is said to be minimal reducing if the only reducing subspaces contained in M are $\{0\}$ and M itself.

2 An equivalence relation

We know that the elements of $\mathcal{T}$ have a specific type of matrix representation with respect to $\{e_i\}_{i\in\mathbb{N}_0}$. Let $T \in \mathcal{T}$ and for $j \in \mathbb{N}_0$ let γ_j denote the non zero entry occurring in the j^{th} column of the matrix of T with respect to $\{e_i\}_{i=0}^{\infty}$. Then there exists a unique bijective map $\psi : \mathbb{N}_0 \to \mathbb{N}_0$ such that γ_j occurs at the $\psi(j)^{th}$ row. Thus if $[a_{i,j}]$ $(i, j \in \mathbb{N}_0)$ denotes the matrix of T with respect to $\{e_i\}_{i=0}^{\infty}$, then

$$a_{i,j} := \begin{cases} \gamma_j, & \text{if } i = \psi(j); \\ 0, & \text{otherwise.} \end{cases}$$

Thus for each $j \in \mathbb{N}_0$, $Te_j = \gamma_j e_{\psi(j)}$. Also $\|T\| = \sup_j |\gamma_j|$.

As we are considering the operator weighted sequence space $l_B^2(K)$, where the uniformly bounded weight sequence $B = \{B_n\}$ is in $\mathcal{T}$, so for each $n \in \mathbb{N}_0$ there exists a unique bijective map ψ_n on $\mathbb{N}_0$ such that $B_n e_j = \gamma_j^{(n)} e_{\psi_n(j)}$, where $\gamma_j^{(n)}$ denotes the unique non zero entry occurring in the j^{th} column of the matrix of B_n.

Definition 2.1. Let $B = \{B_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$, and for each $n \in \mathbb{N}_0$ let $\gamma_j^{(n)}$ denote the unique non zero entry occurring in the j^{th} column of the matrix of B_n. The weight sequence $\{B_n\}$ is said to be of *type* I if for each pair of distinct non negative integers m and n there exist some positive integer k such that $\frac{\gamma_m^{(k)}}{\gamma_m^{(0)}} \neq \frac{\gamma_n^{(k)}}{\gamma_n^{(0)}}$. Otherwise, it is said to be of *type* II. Thus $\{B_n\}$ is of *type* II if there exist distinct non negative integers m and n such that $\frac{\gamma_m^{(k)}}{\gamma_m^{(0)}} = \frac{\gamma_n^{(k)}}{\gamma_n^{(0)}}$ for every positive integer k.

Definition 2.2. Let $B = \{B_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$, and for each $n \in \mathbb{N}_0$ let $\gamma_j^{(n)}$ denote the unique non zero entry occurring in the j^{th} column of the matrix of B_n. Two non negative integers m and n are said to be B-related (denoted by $m \sim^B n$) if for every positive integer k, we have $\frac{\gamma_m^{(k)}}{\gamma_m^{(0)}} = \frac{\gamma_n^{(k)}}{\gamma_n^{(0)}}$. Clearly, $\sim^B$ is an equivalence relation on the set $\mathbb{N}_0$.

Definition 2.3. Let $B = \{B_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$. A weight sequence $\{B_n\}$ of *type* II is said to be of *type* III if $\sim^B$ partitions $\mathbb{N}_0$ into a finite number of equivalence classes.

Remark 2.4. The above definitions are motivated by similar definitions given in [2]. In fact for $dimK = N < \infty$ the two definitions refer to the same idea. In [2] the minimal reducing subspaces of $M_z^N (N > 1)$ on the space $H_w^2 := \{f(z) = \sum_{k=0}^{\infty} a_k z^k : \|f\|_w^2 = \sum w_k |a_k|^2 < \infty\}$ is determined, where $w = \{w_0, w_1, \ldots\}$ is a sequence of positive numbers. If in the present study we consider $dimK = N$, and for each $n \in \mathbb{N}_0$ define $B_n = diag\big(\sqrt{w_{nN}}, \sqrt{w_{nN+1}}, \ldots, \sqrt{w_{(n+1)N-1}}\big)$, then M_z^N on H_w^2 is unitarily equivalent to the unilateral shift S on $l_B^2(K)$.

Definition 2.5. Let $F = \sum_{i\in\mathbb{N}_0} \alpha_i f_{i,0}$ be a non-zero vector in $l_B^2(K)$. The order of F, denoted as $o(F)$, is defined as the smallest non negative integer m such that $\alpha_m \neq 0$.

Definition 2.6. If $f = \sum_{i \in \mathbb{N}_0} \alpha_i e_i$ is a non-zero vector in K, then order of f, denoted as $o(f)$, is defined to be the smallest non negative integer m such that $\alpha_m \neq 0$.

Definition 2.7. Let Y be a non-zero non-empty subset of K. Then order of Y, denoted as $o(Y)$, is defined to be the non negative integer m satisfying the following conditions:
(i) $o(f) \geq m$ for all $f \in Y$, and
(ii) there exists $\tilde{f} \in Y$ such that $o(\tilde{f}) = m$.

Definition 2.8. Let X be a subset of $l^2_B(K)$ and $\mathcal{L}_X := \{f_0 : (f_0, f_1, ...) \in X\}$. If $\mathcal{L}_X$ is a non-zero subset of K, then order of X, denoted as $o(X)$, is defined as $o(\mathcal{L}_X)$.

Definition 2.9. Let $B = \{B_n\}_{n \in \mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$. A linear expression $F = \sum_{i \in \mathbb{N}_0} \alpha_i f_{i,0}$ in $l^2_B(K)$ is said to be B-transparent if for every pair of non-zero scalars α_i and α_j, we have $i \sim^B j$.

Definition 2.10. Let $B = \{B_n\}_{n \in \mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$, and S be the unilateral shift on $l^2_B(K)$. Let $\mathcal{S}$ be the vector space of all finite linear combinations of finite products of S and S^*. For non-zero $F \in l^2_B(K)$, let $\mathcal{S}F := \{TF : T \in \mathcal{S}\}$. Then the closure of $\mathcal{S}F$ in $l^2_B(K)$ is a reducing subspace of S, denoted by X_F. Clearly X_F is the smallest reducing subspace of $l^2_B(K)$ containing F.

Lemma 2.11. *Let $B = \{B_n\}_{n \in \mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$, and for each $n \in \mathbb{N}_0$ let $\gamma_j^{(n)}$ denote the unique non zero entry occurring in the j^{th} column of the matrix of B_n. If S is the unilateral shift on $l^2_B(K)$, then for $i, j \in \mathbb{N}_0$, the following will hold:*

(i) $S^* f_{i,j} = \begin{cases} 0 & \text{if } j = 0, \\ \left| \frac{\gamma_i^{(j)}}{\gamma_i^{(j-1)}} \right| f_{i,j-1} & \text{if } j > 0. \end{cases}$

(ii) For any non negative integer k, $(S^k)^* S^k f_{i,j} = \left| \frac{\gamma_i^{(j+k)}}{\gamma_i^{(j)}} \right|^2 f_{i,j}$.

Proof. (i) For $i \in \mathbb{N}_0$, $\langle SX, f_{i,0} \rangle = 0$ for all $X \in l^2_B(K)$, which implies $S^* f_{i,0} = 0$.
Next let $X = (x_0, x_1, \ldots) \in l^2_B(K)$ and $x_j = \sum_{t=0}^{\infty} \alpha_t^{(j)} e_t$ for each $j \in \mathbb{N}_0$.
Then for $j > 0$, we have $\langle SX, f_{i,j} \rangle = \frac{1}{|\gamma_i^{(j)}|} \langle B_j x_{j-1}, B_j e_i \rangle = \alpha_i^{(j-1)} |\gamma_i^{(j)}|$.
Choosing $\lambda_{i,j} = \left| \frac{\gamma_i^{(j)}}{\gamma_i^{(j-1)}} \right|$, we get

$$\langle X, \lambda_{i,j} f_{i,j-1} \rangle = \frac{\lambda_{i,j}}{|\gamma_i^{(j-1)}|} \langle B_{j-1} x_{j-1}, B_{j-1} e_i \rangle = \frac{\lambda_{i,j}}{|\gamma_i^{(j-1)}|} \alpha_i^{(j-1)} \|B_{j-1} e_i\|^2 = \alpha_i^{(j-1)} |\gamma_i^{(j)}|.$$

Therefore, $\langle SX, f_{i,j}\rangle = \langle X, \lambda_{i,j} f_{i,j-1}\rangle$, and so $S^* f_{i,j} = \lambda_{i,j} f_{i,j-1}$ for $j > 0$.
(ii) As $S^*Sf_{i,j} = \left|\frac{\gamma_i^{(j+1)}}{\gamma_i^{(j)}}\right| S^* f_{i,j+1} = \left|\frac{\gamma_i^{(j+1)}}{\gamma_i^{(j)}}\right|^2 f_{i,j}$, so the result holds for $k = 1$.
Suppose, $(S^*)^n S^n f_{i,j} = \left|\frac{\gamma_i^{(j+n)}}{\gamma_i^{(j)}}\right|^2 f_{i,j}$ holds for $n = k$. We will show that it also holds for $n = k + 1$.

$$\begin{aligned}(S^*)^{k+1} S^{k+1} f_{i,j} =& \left|\frac{\gamma_i^{(j+1)}}{\gamma_i^{(j)}}\right| S^*({S^*}^k S^k) f_{i,j+1} \\ =& \left|\frac{\gamma_i^{(j+1)}}{\gamma_i^{(j)}}\right| \left|\frac{\gamma_i^{(j+1+k)}}{\gamma_i^{(j+1)}}\right|^2 S^* f_{i,j+1} \\ =& \left|\frac{\gamma_i^{(j+1)}}{\gamma_i^{(j)}}\right| \left|\frac{\gamma_i^{(j+1+k)}}{\gamma_i^{(j+1)}}\right|^2 \left|\frac{\gamma_i^{(j+1)}}{\gamma_i^{(j)}}\right| f_{i,j} \\ =& \left|\frac{\gamma_i^{(j+1+k)}}{\gamma_i^{(j)}}\right|^2 f_{i,j}\end{aligned}$$

Thus, the results holds for all k by induction. □

Lemma 2.12. *Let $B = \{B_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$, and for each $n \in \mathbb{N}_0$ let $\gamma_j^{(n)}$ denote the unique non zero entry occurring in the j^{th} column of the matrix of B_n. Let $F = \sum_{i=0}^{\infty} \alpha_i f_{i,0}$ be B-transparent in $l_B^2(K)$ with $o(F) = m$. If for each $k \in \mathbb{N}_0$, $\tilde{F}_k := \left|\frac{\gamma_m^{(k)}}{\gamma_m^{(0)}}\right| \sum_{i=0}^{\infty} \alpha_i f_{i,k}$, then the following will hold:*
(i) $(S^k)^* S^k F = \left|\frac{\gamma_m^{(k)}}{\gamma_m^{(0)}}\right|^2 F$.
(ii) $S\tilde{F}_k = \tilde{F}_{k+1}$ and $S^*\tilde{F}_k = \begin{cases} 0, & \text{if } k = 0; \\ \left|\frac{\gamma_m^{(k)}}{\gamma_m^{(k-1)}}\right|^2 \tilde{F}_{k-1}, & \text{if } k > 0. \end{cases}$
(iii) X_F is the closed linear span of $\{\tilde{F}_k : k \in \mathbb{N}_0\}$.

Proof. Since $o(F) = m$ so $\alpha_i = 0, \forall i < m$. Let $\Lambda = \{i \geq m : \alpha_i \neq 0\}$. Then $m \in \Lambda$, and for $i \in \Lambda$, $\frac{\gamma_i^{(k)}}{\gamma_i^{(0)}} = \frac{\gamma_m^{(k)}}{\gamma_m^{(0)}}$ for each positive integer k.

(i) For $i \in \Lambda$ and positive integer k, by Lemma 2.11(ii) we have
$(S^k)^* S^k f_{i,0} = \left|\frac{\gamma_i^{(k)}}{\gamma_i^{(0)}}\right|^2 f_{i,0} = \left|\frac{\gamma_m^{(k)}}{\gamma_m^{(0)}}\right|^2 f_{i,0}$.
Thus, $(S^k)^* S^k F = (S^k)^* S^k (\sum_{i\in\Lambda} \alpha_i f_{i,0}) = \sum_{i\in\Lambda} \alpha_i \left|\frac{\gamma_m^{(k)}}{\gamma_m^{(0)}}\right|^2 f_{i,0} = \left|\frac{\gamma_m^{(k)}}{\gamma_m^{(0)}}\right|^2 F$.
(ii) For $i, j \in \mathbb{N}_0$, $Sf_{i,j} = \left|\frac{\gamma_i^{(j+1)}}{\gamma_i^{(j)}}\right| f_{i,j+1}$, and so

$$S\tilde{F}_k = \left|\frac{\gamma_m^{(k)}}{\gamma_m^{(0)}}\right| \sum_{i\in\Lambda} \alpha_i\, Sf_{i,k} = \sum_{i\in\Lambda} \alpha_i \left|\frac{\gamma_i^{(k)}}{\gamma_i^{(0)}}\right| \left|\frac{\gamma_i^{(k+1)}}{\gamma_i^{(k)}}\right| f_{i,k+1} = \left|\frac{\gamma_m^{(k+1)}}{\gamma_m^{(0)}}\right| \sum_{i\in\Lambda} \alpha_i f_{i,k+1} = \tilde{F}_{k+1}.$$

As $S^* f_{i,0} = 0$ so $S^* \tilde{F}_0 = 0$.

For $k > 0$, $S^* \tilde{F}_k = \left|\frac{\gamma_m^{(k)}}{\gamma_m^{(0)}}\right| \sum_{i=0}^{\infty} \alpha_i S^* f_{i,k} = \left|\frac{\gamma_m^{(k)}}{\gamma_m^{(0)}}\right| \sum_{i=0}^{\infty} \alpha_i \left|\frac{\gamma_i^{(k)}}{\gamma_i^{(k-1)}}\right| f_{i,k-1} = \left|\frac{\gamma_m^{(k)}}{\gamma_m^{(k-1)}}\right|^2 \tilde{F}_{k-1}$

(iii) By (ii) each $\tilde{F}_k \in X_F$ and so the *closed linear span*$\{\tilde{F}_k : k \in \mathbb{N}_0\}$ is a non-zero reducing subspace of W contained in X_F. Thus, by minimality of X_F, we have $X_F =$ *closed linear span*$\{\tilde{F}_k : k \in \mathbb{N}_0\}$. □

Definition 2.13. Let $B = \{B_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$, and S be the unilateral shift on $l_B^2(K)$. Let $\Omega_1, \Omega_2, ...$ be the disjoint equivalence classes of $\mathbb{N}_0$ under the relation $\sim^B$. Consider $F = \sum_{i=0}^{\infty} \alpha_i f_{i,0}$ in $l_B^2(K)$. For each k, let $q_k := \sum_{i\in\Omega_k} \alpha_i g_{i,0}$. Dropping those q_k which are zero, the remaining q_k's are arranged as $f_1, f_2, ...$ in such a way that for $i < j$ we have $o(f_i) < o(f_j)$. The resulting decomposition $F = f_1 + f_2 + ...$ is called the canonical decomposition of F. Clearly each f_i is B-transparent in $l_B^2(K)$.
If there exists a finite positive integer n such that $F = f_1 + f_2 + \cdots + f_n$, then F in the above case is said to have a finite canonical decomposition.

Lemma 2.14. *Let $B = \{B_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$, and for each $n \in \mathbb{N}_0$ let $\gamma_j^{(n)}$ denote the unique non zero entry occurring in the j^{th} column of the matrix of B_n. Let S be the unilateral shift on $l_B^2(K)$, and X be a reducing subspace of S in $l_B^2(K)$. If $F = \sum_{i\in\mathbb{N}_0} \alpha_i f_{i,0}$ in X has a finite canonical decomposition $F = f_1 + f_2 + \cdots + f_n$, then each f_i is in X_F.*

Proof. Let $o(f_i) = m_i$ so that $m_1 < m_2 < \cdots < m_n$ and no two of them are B-related. For $2 \le i \le n$, as $m_1 \nsim^B m_i$, and so there exists a positive integer k_i such that $\frac{\gamma_{m_1}^{(k_i)}}{\gamma_{m_1}^{(0)}} \ne \frac{\gamma_{m_i}^{(k_i)}}{\gamma_{m_i}^{(0)}}$. Let k_i be the smallest positive integer having this property.

Let $q_1 := F$ and for $2 \le i \le n$, $q_i := \left[\left|\frac{\gamma_{m_i}^{(k_i)}}{\gamma_{m_i}^{(0)}}\right|^2 - (S^{k_i})^* S^{k_i}\right] q_{i-1}$.

Then $q_i \in X_F \; \forall \; 1 \le i \le n$. Also $q_n = (\beta_2 \ldots \beta_n) f_1$, where $\beta_i = \left|\frac{\gamma_{m_i}^{(k_i)}}{\gamma_{m_i}^{(0)}}\right|^2 - \left|\frac{\gamma_{m_1}^{(k_i)}}{\gamma_{m_1}^{(0)}}\right|^2$ for $2 \le i \le n$. As each $\beta_i \ne 0$ so $q_n \in X_F \implies f_1 \in X_F$.
In a similar way it can be shown that $f_2, \ldots, f_n$ are also in X_F. □

3 An Extremal Problem

Theorem 3.1. *Let $B = \{B_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$, and S be the unilateral shift on $l_B^2(K)$. Let X be a non zero reducing* *subspace of S in $l^2_B(K)$ with $o(X) = m$. Then the extremal problem*

$$\sup\{Re\ \alpha_m : F = (f_0, f_1, ...) \in X,\ \|F\| \leq 1,\ f_0 = \sum_{i \in \mathbb{N}_0} \alpha_i e_i.\}$$

has a unique solution $G = \sum_{i \in \mathbb{N}_0} \alpha_i g_{i,0} \in X$ with $\|G\| = 1$ and $o(G) = m$.

Proof. For $F = (f_0, f_1, \ldots) \in X$, we define $\varphi : X \to \mathbb{C}$ as $\varphi(F) = \alpha_m$ where $f_0 = \sum_{i \in \mathbb{N}_0} \alpha_i e_i$. Since $o(X) = m$, so φ is a non zero bounded linear functional on X. From [1], it follows that the extremal problem has a unique solution G in X such that $\|G\| = 1$, $\varphi(G) > 0$ and

$$\begin{aligned} \varphi(G) &= \sup\{Re\ \varphi(F) : F \in X,\ \|F\| \leq 1\} \\ &= \sup\{Re\ \alpha_m : F = (f_0, f_1, ...) \in X,\ \|F\| \leq 1,\ f_0 = \sum_{i \in \mathbb{N}_0} \alpha_i e_i.\} \end{aligned}$$

We claim that G has the form $G = \sum_i \alpha_i f_{i,0}$ with $o(G) = m$.
If $F \in X$ and $\|F\| < 1$, then by maximality of G we must have $Re\varphi(F) < \varphi(G)$. Now as $Re\varphi(G + SF) = \varphi(G)\ \forall\, F \in X$, so we must have $\|G + SF\| \geq 1$. This implies that $G \perp SF\ \forall\, F \in X$. In particular $\langle G, SS^*G\rangle = 0$ which implies that $S^*G = 0$. Thus G is of the form $G = \sum_{i \in \mathbb{N}_0} \alpha_i f_{i,0}$. Also $\varphi(G) > 0$ and $o(X) = m$ together imply $o(G) = m$. □
Note: The function G in Theorem 3.1 will be called the *extremal function* of X.

Theorem 3.2. *Let $B = \{B_n\}_{n \in \mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$, and S be the unilateral shift on $l^2_B(K)$. If the extremal function of a non-zero reducing subspace of S in $l^2_B(K)$ has a finite canonical decomposition, then it must be B-transparent.*

Proof. Let X be a non-zero reducing subspace of S in $l^2_B(K)$ and $G = \sum_i \alpha_i f_{i,0}$ be its extremal function with $o(G) = m$. Let $G = g_1 + g_2 + \cdots + g_n$ be the finite canonical decomposition of G. Each g_i is B-transparent and also by Lemma 2.14, each of them is in X_G. Clearly $o(g_1) = m$ and $\|g_1\| \leq \|G\| = 1$. So by extremality of G, we must have $G = g_1$. Thus G is B-transparent. □

4 Minimal reducing subspaces

Theorem 4.1. *Let $B = \{B_n\}_{n \in \mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$, and for each $n \in \mathbb{N}_0$ let $\gamma_j^{(n)}$ denote the unique non zero entry occurring in the j^{th} column of the matrix of B_n. Also let S be the unilateral shift on $l^2_B(K)$. If X is a minimal reducing subspace of S in $l^2_B(K)$ and $F = \sum_i \alpha_i f_{i,0}$ is in X, then F is B-transparent.*

Proof. Let $o(F) = m$ and, if possible F is not B-transparent. So we must have a positive integer $k > m$ such that $\alpha_k \neq 0$ and $k \nsim^B m$. This means that there exists a positive integer l such that $\frac{\gamma_k^{(l)}}{\gamma_k^{(0)}} \neq \frac{\gamma_m^{(l)}}{\gamma_m^{(0)}}$.

We define $G := (S^l)^* S^l F - \left|\frac{\gamma_m^{(l)}}{\gamma_m^{(0)}}\right|^2 F$. Clearly, G is in X, and we get

$$\begin{aligned}
G =& (S^l)^* S^l F - \left|\frac{\gamma_m^{(l)}}{\gamma_m^{(0)}}\right|^2 F \\
=& (S^l)^* S^l (\sum_{i=m}^{\infty} \alpha_i f_{i,0}) - \left|\frac{\gamma_m^{(l)}}{\gamma_m^{(0)}}\right|^2 (\sum_{i=m}^{\infty} \alpha_i f_{i,0}) \\
=& \sum_{i=m}^{\infty} \alpha_i \left|\frac{\gamma_i^{(l)}}{\gamma_i^{(0)}}\right|^2 f_{i,0} - \sum_{i=m}^{\infty} \alpha_i \left|\frac{\gamma_m^{(l)}}{\gamma_m^{(0)}}\right|^2 f_{i,0} \\
=& \sum_{i=m+1}^{\infty} \alpha_i \left[\left|\frac{\gamma_i^{(l)}}{\gamma_i^{(0)}}\right|^2 - \left|\frac{\gamma_m^{(l)}}{\gamma_m^{(0)}}\right|^2\right] f_{i,0}
\end{aligned}$$

Thus, $G = \sum_{i=m+1}^{\infty} \gamma_i f_{i,0}$, where $\gamma_i = \alpha_i \left[\left|\frac{\gamma_i^{(l)}}{\gamma_i^{(0)}}\right|^2 - \left|\frac{\gamma_m^{(l)}}{\gamma_m^{(0)}}\right|^2\right]$. Also since $\gamma_k \neq 0$ so $G \neq 0$. Moreover, $o(F) < o(G) \implies F \notin X_G$. Hence X_G is a non-zero reducing subspace properly contained in X which contradicts the minimality of the X. Hence F must be B-transparent. □

As an immediate corollary of the above result we have the following :

Corollary 4.2. *The extremal function of a minimal reducing subspace of S in $l_B^2(K)$ is always B-transparent.*

Theorem 4.3. *Let $B = \{B_n\}_{n \in \mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$, and S be the unilateral shift on $l_B^2(K)$. Let X be a reducing subspace of S in $l_B^2(K)$. Then X is minimal if and only if $X = X_F$ where F is B-transparent.*

Proof. If X is minimal and G is the associated extremal function, then the reducing subspace $X_G \subseteq X$. The minimality of X gives $X = X_G$. Note that by Corollary 4.2, G is B-transparent.

Conversely, let $X = X_F$, where F is B-transparent. Clearly X_F is a reducing subspace. We claim that X_F is minimal. Let Y be a non zero reducing subspace of S contained in X_F and H be its extremal function, which is transparent. Then $H \in X_F$ and so by Lemma 2.12(i), H is a scalar multiple of F. In particular, $F \in Y$. Thus, $Y = X_F$ which means that X_F must be minimal. □

Corollary 4.4. *Every reducing subspace of S in $l_B^2(K)$, whose extremal function has a finite canonical decomposition, contains a minimal reducing subspace.*

Proof. The proof follows immediately from Theorem 3.2 and Theorem 4.3. □

5 Conclusion

Theorem 5.1. *Let $B = \{B_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$, and S be the unilateral shift on $l^2_B(K)$. If the weight sequence $\{B_n\}_{n\in\mathbb{N}_0}$ is of type I, then $X_{f_{n,0}}$ for $n \in \mathbb{N}_0$ are the only minimal reducing subspaces of S in $l^2_B(K)$.*

Proof. Let X be a minimal reducing subspace of S and G be its extremal function so that $X = X_G$. Since the weight sequence $\{B_n\}_{n\in\mathbb{N}_0}$ is of *type* I, so the only transparent functions are $f_{n,0}$ for $n \in \mathbb{N}_0$ and their scalar multiples. The result now follows from Theorem 4.3. □

Theorem 5.2. *Let $B = \{B_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$, and S be the unilateral shift on $l^2_B(K)$. If $\{B_n\}_{n\in\mathbb{N}_0}$ is of type II, then S has minimal reducing subspaces other than $X_{f_{n,0}}$, $n \in \mathbb{N}_0$.*

Proof. Since the weight sequence $\{B_n\}_{n\in\mathbb{N}_0}$ is of *type* II, so we can form a transparent function $F = \sum_{i\in\mathbb{N}_0} \alpha_i f_{i,0}$ where more than one α_i's are non zero. Clearly, X_F is a minimal reducing subspace of S in $l^2_B(K)$ such that $X_F \neq X_{f_{n,0}}$ for any $n \in \mathbb{N}_0$. □

Theorem 5.3. *Let $B = \{B_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$, and S be the unilateral shift on $l^2_B(K)$. If $\{B_n\}_{n\in\mathbb{N}_0}$ is of type III, then every reducing subspace of S in $l^2_B(K)$ must contain a minimal reducing subspace.*

Proof. Let X be a reducing subspace of S and G be its extremal function. Since the weight sequence $\{B_n\}_{n\in\mathbb{N}_0}$ is of *type* III, so G must have a finite canonical decomposition, say $g_1 + g_2 + ... + g_n$. By Lemma 2.14, for each $1 \leq i \leq n$, $g_i \in X$ and so each X_{g_i} is a minimal reducing subspace of S in $l^2_B(K)$ contained in X. □

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