East-West J. of Mathematics: Vol. 19, No 1 (2017) pp. 52-64

THE RECIPROCAL SUMS OF EVEN AND ODD TERMS IN THE PELL SEQUENCE

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Abstract

The reciprocal sums of even and odd terms in the Pell sequence are considered. New interesting identities involving the partial finite sums of the even-indexed and the odd-indexed reciprocal Pell numbers are derived.

1 Introduction

The *Pell sequence* is a sequence of integers satisfying the second-order linear recurrence relation

$$P_{n+2} = 2P_{n+1} + P_n, \quad P_0 = 0,\ P_1 = 1.$$

The integer P_n is called the nth *Pell number*. Solving this recurrence gives us a simple formula for the nth Pell number as

$$P_n = \frac{1}{2\sqrt{2}}\left[\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n\right].$$

The Pell sequence is closely related to the Fibonacci sequence, which is defined by

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0,\ F_1 = 1.$$

The integer F_n is called the nth *Fibonacci number*. The reciprocal sums of the Fibonacci numbers and the Pell numbers have been studied increasingly over a decade; see for example [1 - 9]. In 2009, Ohtsuka and Nakamura [5] established some results about the infinite sums of the reciprocal Fibonacci numbers.

*Corresponding author. **Key words:** Pell numbers, reciprocal sum
2010 AMS Mathematics classification: 11B39

Theorem 1.1. *For all $n \geq 2$,*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2}, & \text{if } n \text{ is even;} \\ F_{n-2} - 1, & \text{if } n \text{ is odd,} \end{cases} \tag{1.1}$$

where $\lfloor \cdot \rfloor$ denotes the floor function.

Holliday and Komatsu [2], and independently, Zhang ang Wang [9] also proved interesting identities for the infinite sums of the Pell numbers.

Theorem 1.2.

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{P_k} \right)^{-1} \right\rfloor = \begin{cases} P_{n-1} + P_{n-2}, & \text{if } n \text{ is even and } n \geq 2; \\ P_{n-1} + P_{n-2} - 1, & \text{if } n \text{ is odd and } n \geq 1, \end{cases}$$

provided that $P_{-1} = P_1 = 1$.

Wang and Wen [6] strengthened (1.1) to finite partial sums.

Theorem 1.3. *(i) For all $n \geq 4$,*

$$\left\lfloor \left(\sum_{k=n}^{2n} \frac{1}{F_k} \right)^{-1} \right\rfloor = F_{n-2}.$$

(ii) If $m \geq 3$ and $n \geq 2$, then

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2}, & \text{if } n \text{ is even;} \\ F_{n-2} - 1, & \text{if } n \text{ is odd.} \end{cases}$$

Recently, Wang and Zhang [7] have obtained some interesting results about the reciprocal sums of the Fibonacci numbers with even or odd indexes.

Theorem 1.4. *We have*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_{2k}} \right)^{-1} \right\rfloor = \begin{cases} F_{2n-1}, & \text{if } m = 2 \text{ and } n \geq 3; \\ F_{2n-1} - 1, & \text{if } m \geq 3 \text{ and } n \geq 1. \end{cases} \tag{1.2}$$

Theorem 1.5. *For all $n \geq 1$ and $m \geq 2$, we have*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_{2k-1}} \right)^{-1} \right\rfloor = F_{2n-2}. \tag{1.3}$$

It is thus natural to ask whether there exist similar formulae for the Pell sequence. In 2017, Choo [1] proved, among other things, the following identities related to the finite sums of reciprocal Pell numbers. If $n \geq 2$, then

$$\left\lfloor \left(\sum_{k=n}^{2n} \frac{1}{P_k} \right)^{-1} \right\rfloor = P_n - P_{n-1}. \tag{1.4}$$

If $n \geq 2$ with n even and $m \geq 2n$, then

$$\left\lfloor \left(\sum_{k=n}^{m} \frac{1}{P_k} \right)^{-1} \right\rfloor = P_n - P_{n-1}. \tag{1.5}$$

If $n \geq 1$ with n odd and $m \geq 3n$, then

$$\left\lfloor \left(\sum_{k=n}^{m} \frac{1}{P_k} \right)^{-1} \right\rfloor = P_n - P_{n-1} - 1. \tag{1.6}$$

Unlike the results of Choo, we investigate here the partial finite sums of reciprocal Pell numbers of odd and even indexes. Although, the main results proved here and those of Choo seem very close and they are proved by similar elementary methods, they seem to be quite independent of one another.

2 Basic identities

In this section, we collect some identities involving the Pell numbers that will be used in our main results. They are similar to identities of the Fibonacci numbers and can be proved by induction. Since the proofs are similar, we only give detailed proofs for Lemmas 2.1, 2.3, and 2.6.

Lemma 2.1. *For any positive integer n, we have*

$$P_n^2 - P_{n-1}P_{n+1} = (-1)^{n-1}. \tag{2.1}$$

Proof. We proceed by induction on n. It is clearly true for $n = 1$. Assuming the result holds for any positive integer k, we will show that the equation (2.1) is true for $k + 1$. We have

$$\begin{aligned}
P_{k+1}^2 - P_k P_{k+2} &= (2P_k + P_{k-1})^2 - P_k (2P_{k+1} + P_k) \\
&= 4P_k^2 + 4P_{k-1}P_k + P_{k-1}^2 - 2P_k P_{k+1} - P_k^2 \\
&= 3P_k^2 + P_{k-1}(2P_k + P_{k-1}) - 2P_k (P_{k+1} - P_{k-1}) \\
&= 3P_k^2 + P_{k-1}P_{k+1} - 4P_k^2 \\
&= -P_k^2 + P_{k-1}P_{k+1} \\
&= (-1)^k.
\end{aligned}$$

□

Lemma 2.2. *For any positive integer $n \geq 2$, we have*

$$P_n^2 - P_{n-2}P_{n+2} = 4(-1)^n. \tag{2.2}$$

Lemma 2.3. *For any positive integers a and b, we have*

$$P_aP_b + P_{a+1}P_{b+1} = P_{a+b+1}. \tag{2.3}$$

Proof. Let a be a positive integer. We proceed by induction on b. For $b = 1$, we have

$$P_aP_1 + P_{a+1}P_2 = P_a + 2P_{a+1} = P_{a+2}.$$

Now, let k be any positive integer. Assume that the equation (2.3) holds for any positive integer $b \leq k$. We get

$$\begin{aligned} P_aP_{k+1} + P_{a+1}P_{k+2} &= P_a\left(2P_k + P_{k-1}\right) + P_{a+1}\left(2P_{k+1} + P_k\right) \\ &= 2P_aP_k + P_aP_{k-1} + 2P_{a+1}P_{k+1} + P_{a+1}P_k \\ &= 2\left(P_aP_k + P_{a+1}P_{k+1}\right) + \left(P_aP_{k-1} + P_{a+1}P_k\right) \\ &= 2P_{a+k+1} + P_{a+k} \\ &= P_{a+k+2}. \end{aligned}$$

□

Remark 2.4. Replacing a by $a - 1$ in (2.3), we get

$$P_{a-1}P_b + P_aP_{b+1} = P_{a+b},$$

which implies that

$$P_{a+b} \geq P_aP_{b+1} \geq P_aP_b. \tag{2.4}$$

Lemma 2.5. *For any positive integer n, we have*

$$2P_{2n+1} = P_{n+1}P_{n+2} - P_{n-1}P_n. \tag{2.5}$$

Proof. This follows from setting $a = n - 1$ and $b = n + 1$ in (2.3) and straightforward calculation. □

Lemma 2.6. *Let a and b be two integers with $a \geq b \geq 0$. If $n > a$, then*

$$P_{n+a}P_{n-a-1} - P_{n+b}P_{n-b-1} = (-1)^{n-a}P_{a+b+1}P_{a-b}. \tag{2.6}$$

Proof. We proceed by induction on n. For $n = a + 1$, we have

$$\begin{aligned} P_{(a+1)+a}P_{(a+1)-a-1} - P_{(a+1)+b}P_{(a+1)-b-1} &= P_{2a+1}P_0 - P_{a+b+1}P_{a-b} \\ &= -P_{a+b+1}P_{a-b} \\ &= (-1)^{(a+1)-a}P_{a+b+1}P_{a-b}. \end{aligned}$$

Assume that the result holds for $n > a$. Applying (2.3) twice and using the induction hypothesis, we get

$$\begin{aligned}
&P_{(n+1)+a}P_{(n+1)-a-1} - P_{(n+1)+b}P_{(n+1)-b-1} \\
&\quad = P_{n+a+1}P_{n-a} - P_{n+b+1}P_{n-b} \\
&\quad = (P_{n+a+1}P_{n-a} + P_{n+a}P_{n-a-1}) - P_{n+b+1}P_{n-b} - P_{n+a}P_{n-a-1} \\
&\quad = P_{2n} - P_{n+b+1}P_{n-b} - P_{n+a}P_{n-a-1} \\
&\quad = P_{n+b}P_{n-b-1} - P_{n+a}P_{n-a-1} \\
&\quad = (-1)^{(n+1)-a}P_{a+b+1}P_{a-b}.
\end{aligned}$$

□

Before ending this section, we establish an inequality which forms part of the proof of one of our main results.

Lemma 2.7. *Let n be a positive integer. If $n \geq 3$, then*

$$\frac{1}{P_{4n+2} - P_{4n}} > \sum_{k=n}^{2n} \frac{4}{P_{2k}\left(P_{2k} - P_{2k-2}\right)\left(P_{2k+2} - P_{2k}\right)}. \tag{2.7}$$

Proof. Setting $a = 2$ and $b = 0$, and replacing n by $2n$ in (2.6), we get

$$P_{2n-3}P_{2n+2} - P_{2n-1}P_{2n} = 10. \tag{2.8}$$

Applying (2.5) and (2.8), we obtain

$$\begin{aligned}
P_{2n-3}\left(P_{4n+2} - P_{4n}\right) &= 2P_{2n-3}P_{4n+1} \\
&= P_{2n-3}\left(P_{2n+1}P_{2n+2} - P_{2n-1}P_{2n}\right) \\
&= P_{2n+1}\left(P_{2n-1}P_{2n} + 10\right) - P_{2n-3}P_{2n-1}P_{2n} \\
&= P_{2n-1}P_{2n}P_{2n+1} - \left(P_{2n-3}P_{2n-1}P_{2n} - 10P_{2n+1}\right).
\end{aligned}$$

Setting $a = 1$ and $b = 0$, and replacing n by $2n$ in (2.6), we get

$$P_{2n-2}P_{2n+1} - P_{2n-1}P_{2n} = -2. \tag{2.9}$$

For $n \geq 3$, we know that $P_{2n-3} \geq 5$ and $P_{2n-2} \geq 12$. Then applying (2.9), we obtain

$$P_{2n-3}P_{2n-1}P_{2n} = P_{2n-3}\left(P_{2n-2}P_{2n+1} + 2\right) \geq 60P_{2n+1} + 10 > 10P_{2n+1}.$$

Therefore,

$$4P_{2n-3}\left(P_{4n+2} - P_{4n}\right) < 4P_{2n-1}P_{2n}P_{2n+1} = P_{2n}\left(P_{2n} - P_{2n-2}\right)\left(P_{2n+2} - P_{2n}\right),$$

which is equivalent to

$$\frac{4\left(P_{4n+2}-P_{4n}\right)}{P_{2n}\left(P_{2n}-P_{2n-2}\right)\left(P_{2n+2}-P_{2n}\right)}<\frac{1}{P_{2n-3}}.$$

Hence,

$$\begin{aligned}
&\frac{1}{P_{4n+2}-P_{4n}}-\sum_{k=n}^{2n}\frac{4}{P_{2k}\left(P_{2k}-P_{2k-2}\right)\left(P_{2k+2}-P_{2k}\right)}\\
&\quad=\frac{1}{P_{4n+2}-P_{4n}}-\frac{1}{P_{4n+2}-P_{4n}}\sum_{k=n}^{2n}\frac{4\left(P_{4n+2}-P_{4n}\right)}{P_{2k}\left(P_{2k}-P_{2k-2}\right)\left(P_{2k+2}-P_{2k}\right)}\\
&\quad\geq\frac{1}{P_{4n+2}-P_{4n}}-\frac{1}{P_{4n+2}-P_{4n}}\sum_{k=n}^{2n}\frac{4\left(P_{4n+2}-P_{4n}\right)}{P_{2n}\left(P_{2n}-P_{2n-2}\right)\left(P_{2n+2}-P_{2n}\right)}\\
&\quad>\frac{1}{P_{4n+2}-P_{4n}}-\frac{1}{P_{4n+2}-P_{4n}}\sum_{k=n}^{2n}\frac{1}{P_{2n-3}}\\
&\quad=\frac{1}{P_{4n+2}-P_{4n}}\left(\frac{P_{2n-3}-n-1}{P_{2n-3}}\right).
\end{aligned}$$

It is not hard to see that $P_{2n-3}>n+1$ for $n\geq 3$, which completes the proof. □

3 Main Results

We begin this section with the partial finite sums of even-indexed reciprocal Pell numbers.

Theorem 3.1. *For all positive integers $n\geq 3$, we have*

$$\left\lfloor\left(\sum_{k=n}^{2n}\frac{1}{P_{2k}}\right)^{-1}\right\rfloor=P_{2n}-P_{2n-2}. \tag{3.1}$$

Proof. Equation (3.1) is equivalent to

$$P_{2n}-P_{2n-2}\leq\left(\sum_{k=n}^{2n}\frac{1}{P_{2k}}\right)^{-1}<P_{2n}-P_{2n-2}+1,$$

or

$$\frac{1}{P_{2n}-P_{2n-2}+1}<\sum_{k=n}^{2n}\frac{1}{P_{2k}}\leq\frac{1}{P_{2n}-P_{2n-2}}. \tag{3.2}$$

By elementary calculation and (2.2), for $k \geq 1$ we have

$$\begin{aligned}
&\frac{1}{P_{2k}-P_{2k-2}+1}-\frac{1}{P_{2k}}-\frac{1}{P_{2k+2}-P_{2k}+1}\\
&\quad=\frac{P_{2k}\left(P_{2k+2}-2P_{2k}+P_{2k-2}\right)-\left(P_{2k}-P_{2k-2}+1\right)\left(P_{2k+2}-P_{2k}+1\right)}{P_{2k}\left(P_{2k}-P_{2k-2}+1\right)\left(P_{2k+2}-P_{2k}+1\right)}\\
&\quad=\frac{-P_{2k}^2+P_{2k-2}P_{2k+2}+P_{2k-2}-P_{2k+2}-1}{P_{2k}\left(P_{2k}-P_{2k-2}+1\right)\left(P_{2k+2}-P_{2k}+1\right)}\\
&\quad=\frac{-4(-1)^{2k}+P_{2k-2}-P_{2k+2}-1}{P_{2k}\left(P_{2k}-P_{2k-2}+1\right)\left(P_{2k+2}-P_{2k}+1\right)}\\
&\quad=\frac{P_{2k-2}-P_{2k+2}-5}{P_{2k}\left(P_{2k}-P_{2k-2}+1\right)\left(P_{2k+2}-P_{2k}+1\right)}\\
&\quad=-\frac{1}{P_{2k}}\left(\frac{1}{P_{2k}-P_{2k-2}+1}+\frac{1}{P_{2k+2}-P_{2k}+1}\right)\\
&\quad-\frac{3}{P_{2k}\left(P_{2k}-P_{2k-2}+1\right)\left(P_{2k+2}-P_{2k}+1\right)}.
\end{aligned}$$

Summing from n to $2n$ gives

$$\begin{aligned}
&\frac{1}{P_{2n}-P_{2n-2}+1}-\sum_{k=n}^{2n}\frac{1}{P_{2k}}-\frac{1}{P_{4n+2}-P_{4n}+1}\\
&\quad=\sum_{k=n}^{2n}-\frac{1}{P_{2k}}\left(\frac{1}{P_{2k}-P_{2k-2}+1}+\frac{1}{P_{2k+2}-P_{2k}+1}\right)\\
&\quad-\frac{3}{P_{2k}\left(P_{2k}-P_{2k-2}+1\right)\left(P_{2k+2}-P_{2k}+1\right)}.
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{k=n}^{2n}\frac{1}{P_{2k}}&=\frac{1}{P_{2n}-P_{2n-2}+1}-\frac{1}{P_{4n+2}-P_{4n}+1}\\
&+\sum_{k=n}^{2n}\frac{1}{P_{2k}}\left(\frac{1}{P_{2k}-P_{2k-2}+1}+\frac{1}{P_{2k+2}-P_{2k}+1}\right)\\
&+\frac{3}{P_{2k}\left(P_{2k}-P_{2k-2}+1\right)\left(P_{2k+2}-P_{2k}+1\right)}\\
&>\frac{1}{P_{2n}-P_{2n-2}+1}-\frac{1}{P_{4n+2}-P_{4n}+1}+\frac{1}{P_{2n}\left(P_{2n}-P_{2n-2}+1\right)}.
\end{aligned}$$

It follows from (2.3) that

$$\begin{aligned} P_{4n+2} - P_{4n} + 1 &> P_{4n+2} - P_{4n} \\ &= (P_{2n}P_{2n+1} + P_{2n+1}P_{2n+2}) - (P_{2n-1}P_{2n} + P_{2n}P_{2n+1}) \\ &= P_{2n+1}P_{2n+2} - P_{2n-1}P_{2n} \\ &> P_{2n}\left(P_{2n+2} - P_{2n-1}\right) \\ &= P_{2n}\left(2P_{2n+1} + P_{2n} - P_{2n-1}\right) \\ &> P_{2n}\left(P_{2n} - P_{2n-2} + 1\right), \end{aligned}$$

which implies that

$$\frac{1}{P_{4n+2} - P_{4n} + 1} < \frac{1}{P_{2n}\left(P_{2n} - P_{2n-2} + 1\right)}.$$

Therefore,

$$\sum_{k=n}^{2n} \frac{1}{P_{2k}} > \frac{1}{P_{2n} - P_{2n-2} + 1}. \tag{3.3}$$

Again, by elementary calculation and (2.2), for $k \geq 1$ we have

$$\begin{aligned} &\frac{1}{P_{2k} - P_{2k-2}} - \frac{1}{P_{2k}} - \frac{1}{P_{2k+2} - P_{2k}} \\ &\quad = \frac{P_{2k}\left(P_{2k+2} - 2P_{2k} + P_{2k-2}\right) - \left(P_{2k} - P_{2k-2}\right)\left(P_{2k+2} - P_{2k}\right)}{P_{2k}\left(P_{2k} - P_{2k-2}\right)\left(P_{2k+2} - P_{2k}\right)} \\ &\quad = \frac{P_{2k-2}P_{2k+2} - P_{2k}^2}{P_{2k}\left(P_{2k} - P_{2k-2}\right)\left(P_{2k+2} - P_{2k}\right)} \\ &\quad = \frac{-4}{P_{2k}\left(P_{2k} - P_{2k-2}\right)\left(P_{2k+2} - P_{2k}\right)}. \end{aligned} \tag{3.4}$$

Summing from n to $2n$ gives

$$\frac{1}{P_{2n} - P_{2n-2}} - \sum_{k=n}^{2n} \frac{1}{P_{2k}} - \frac{1}{P_{4n+2} - P_{4n}} = \sum_{k=n}^{2n} \frac{-4}{P_{2k}\left(P_{2k} - P_{2k-2}\right)\left(P_{2k+2} - P_{2k}\right)}.$$

Then from (2.7), we get

$$\begin{aligned} \sum_{k=n}^{2n} \frac{1}{P_{2k}} &= \frac{1}{P_{2n} - P_{2n-2}} - \frac{1}{P_{4n+2} - P_{4n}} + \sum_{k=n}^{2n} \frac{4}{P_{2k}\left(P_{2k} - P_{2k-2}\right)\left(P_{2k+2} - P_{2k}\right)} \\ &< \frac{1}{P_{2n} - P_{2n-2}}. \end{aligned} \tag{3.5}$$

Combing (3.3) and (3.5), the inequalities in (3.2) follow. □

Theorem 3.2. *Let m and n be positive integers. If $m \geq 3$ and $n \geq 1$, we have*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{P_{2k}} \right)^{-1} \right\rfloor = P_{2n} - P_{2n-2} - 1. \tag{3.6}$$

Proof. Equation (3.6) is equivalent to

$$P_{2n} - P_{2n-2} - 1 \leq \left(\sum_{k=n}^{mn} \frac{1}{P_{2k}} \right)^{-1} < P_{2n} - P_{2n-2},$$

or

$$\frac{1}{P_{2n} - P_{2n-2}} < \sum_{k=n}^{mn} \frac{1}{P_{2k}} \leq \frac{1}{P_{2n} - P_{2n-2} - 1}. \tag{3.7}$$

By elementary calculation and (2.2), for $k \geq 1$, we get

$$\begin{aligned}
&\frac{1}{P_{2k} - P_{2k-2} - 1} - \frac{1}{P_{2k}} - \frac{1}{P_{2k+2} - P_{2k} - 1} \\
&\quad = \frac{P_{2k}\left(P_{2k+2} - 2P_{2k} + P_{2k-2}\right) - \left(P_{2k} - P_{2k-2} - 1\right)\left(P_{2k+2} - P_{2k} - 1\right)}{P_{2k}\left(P_{2k} - P_{2k-2} - 1\right)\left(P_{2k+2} - P_{2k} - 1\right)} \\
&\quad = \frac{-P_{2k}^2 + P_{2k-2}P_{2k+2} - P_{2k-2} + P_{2k+2} - 1}{P_{2k}\left(P_{2k} - P_{2k-2} - 1\right)\left(P_{2k+2} - P_{2k} - 1\right)} \\
&\quad = \frac{P_{2k+2} - P_{2k-2} - 5}{P_{2k}\left(P_{2k} - P_{2k-2} - 1\right)\left(P_{2k+2} - P_{2k} - 1\right)}.
\end{aligned}$$

Since

$$P_{2k+2} = 2P_{2k+1} + P_{2k} = 4P_{2k} + 4P_{2k-1} + P_{2k-2} > P_{2k-2} + 5,$$

we have

$$\frac{1}{P_{2k} - P_{2k-2} - 1} - \frac{1}{P_{2k}} - \frac{1}{P_{2k+2} - P_{2k} - 1} > 0. \tag{3.8}$$

Summing from n to mn gives

$$\frac{1}{P_{2n} - P_{2n-2} - 1} - \sum_{k=n}^{mn} \frac{1}{P_{2k}} - \frac{1}{P_{2mn+2} - P_{2mn} - 1} > 0,$$

which implies that

$$\begin{aligned}
\sum_{k=n}^{mn} \frac{1}{P_{2k}} &< \frac{1}{P_{2n} - P_{2n-2} - 1} - \frac{1}{P_{2mn+2} - P_{2mn} - 1} \\
&< \frac{1}{P_{2n} - P_{2n-2} - 1}.
\end{aligned} \tag{3.9}$$

On the other hand, it follows from (3.4) that

$$\frac{1}{P_{2n}-P_{2n-2}}-\sum_{k=n}^{mn}\frac{1}{P_{2k}}-\frac{1}{P_{2mn+2}-P_{2mn}}=\sum_{k=n}^{mn}\frac{-4}{P_{2k}\left(P_{2k}-P_{2k-2}\right)\left(P_{2k+2}-P_{2k}\right)}.$$

Then

$$\begin{aligned}\sum_{k=n}^{mn}\frac{1}{P_{2k}}&=\frac{1}{P_{2n}-P_{2n-2}}-\frac{1}{P_{2mn+2}-P_{2mn}}+\sum_{k=n}^{mn}\frac{4}{P_{2k}\left(P_{2k}-P_{2k-2}\right)\left(P_{2k+2}-P_{2k}\right)}\\&>\frac{1}{P_{2n}-P_{2n-2}}-\frac{1}{P_{2mn+2}-P_{2mn}}+\frac{1}{P_{2n}\left(P_{2n}-P_{2n-2}\right)\left(P_{2n+2}-P_{2n}\right)}.\end{aligned}$$

For $m \geq 3$, applying (2.4) gives

$$2P_{2mn+1}\geq 2P_{6n+1}\geq 4P_{6n}\geq 4P_{4n}P_{2n+1}\geq 4P_{2n}P_{2n}P_{2n+1}\geq 4P_{2n-1}P_{2n}P_{2n+1},$$

and so

$$P_{2mn+2}-P_{2mn}\geq P_{2n}\left(P_{2n}-P_{2n-2}\right)\left(P_{2n+2}-P_{2n}\right),$$

which is implies that

$$\frac{1}{P_{2n}\left(P_{2n}-P_{2n-2}\right)\left(P_{2n+2}-P_{2n}\right)}\geq\frac{1}{P_{2mn+2}-P_{2mn}}. \tag{3.10}$$

Thus,

$$\sum_{k=n}^{mn}\frac{1}{P_{2k}}>\frac{1}{P_{2n}-P_{2n-2}}. \tag{3.11}$$

Combing (3.9) and (3.11) yields the desired result (3.7). □

Corollary 3.3. *For any positive integer $n \geq 1$, we have*

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{P_{2k}}\right)^{-1}\right\rfloor=P_{2n}-P_{2n-2}-1. \tag{3.12}$$

Proof. By (3.7) and the monotone convergence theorem, $\sum_{k=n}^{\infty}\frac{1}{P_{2k}}$ converges and (3.7) also holds when we replace $\sum_{k=n}^{mn}\frac{1}{P_{2k}}$ by $\sum_{k=n}^{\infty}\frac{1}{P_{2k}}$. □

Remark 3.4. Note that $P_{2n}-P_{2n-2}=2P_{2n-1}$. Hence, equations (3.1), (3.6), and (3.12) can be rewritten respectively as

$$\left\lfloor \left(\sum_{k=n}^{2n} \frac{1}{P_{2k}} \right)^{-1} \right\rfloor = 2P_{2n-1},$$

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{P_{2k}} \right)^{-1} \right\rfloor = 2P_{2n-1} - 1,$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{P_{2k}} \right)^{-1} \right\rfloor = 2P_{2n-1} - 1.$$

Our results for the Pell numbers are thus similar to (1.2) of the Fibonacci sequence.

Now, we will consider the partial finite sums of odd-indexed reciprocal Pell numbers.

Theorem 3.5. *Let m and n be positive integers. If $m \geq 2$ and $n \geq 2$, we have*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{P_{2k-1}} \right)^{-1} \right\rfloor = 2P_{2n-2} = P_{2n-1} - P_{2n-3}. \tag{3.13}$$

Proof. Equation (3.13) is equivalent to

$$2P_{2n-2} \leq \left(\sum_{k=n}^{mn} \frac{1}{P_{2k-1}} \right)^{-1} < 2P_{2n-2} + 1,$$

or

$$\frac{1}{2P_{2n-2}+1} < \sum_{k=n}^{mn} \frac{1}{P_{2k-1}} \leq \frac{1}{2P_{2n-2}}. \tag{3.14}$$

By elementary calculation and (2.1), for $k \geq 2$ we get

$$\begin{aligned}
&\frac{1}{2P_{2k-2}+1} - \frac{1}{P_{2k-1}} - \frac{1}{2P_{2k}+1} \\
&\qquad = \frac{P_{2k-1}\left(2P_{2k} - 2P_{2k-2}\right) - \left(2P_{2k-2}+1\right)\left(2P_{2k}+1\right)}{P_{2k-1}\left(2P_{2k-2}+1\right)\left(2P_{2k}+1\right)} \\
&\qquad = \frac{4P_{2k-1}^2 - 4P_{2k-2}P_{2k} - 2P_{2k-2} - 2P_{2k} - 1}{P_{2k-1}\left(2P_{2k-2}+1\right)\left(2P_{2k}+1\right)} \\
&\qquad = \frac{-2P_{2k-2} - 2P_{2k} + 3}{P_{2k-1}\left(2P_{2k-2}+1\right)\left(2P_{2k}+1\right)}.
\end{aligned}$$

Summing from n to mn gives

$$\frac{1}{2P_{2n-2}+1}-\sum_{k=n}^{mn}\frac{1}{P_{2k-1}}-\frac{1}{2P_{2mn}+1}=\sum_{k=n}^{mn}\frac{-2P_{2k-2}-2P_{2k}+3}{P_{2k-1}(2P_{2k-2}+1)(2P_{2k}+1)}.$$

Then

$$\begin{aligned}\sum_{k=n}^{mn}\frac{1}{P_{2k-1}}&=\frac{1}{2P_{2n-2}+1}-\frac{1}{2P_{2mn}+1}+\sum_{k=n}^{mn}\frac{2P_{2k-2}+2P_{2k}-3}{P_{2k-1}(2P_{2k-2}+1)(2P_{2k}+1)}\\&>\frac{1}{2P_{2n-2}+1}-\frac{1}{2P_{2mn}+1}+\frac{2P_{2n-2}}{P_{2n-1}(2P_{2n-2}+1)(2P_{2n}+1)}.\end{aligned}$$

For $m\geq 2$ and $n\geq 2$, applying (2.4) gives

$$\begin{aligned}2P_{2n-2}(2P_{2mn}+1)&>4P_{2n-2}P_{2mn}\\&\geq 4P_{2n-2}P_{4n}\\&\geq 4P_{2n-2}P_{2n}P_{2n+1}=4P_{2n-2}P_{2n}(2P_{2n}+P_{2n-1})\\&\geq 4P_{2n-2}P_{2n-1}P_{2n}+2P_{2n-2}P_{2n-1}+2P_{2n-1}P_{2n}+P_{2n-1}\\&=P_{2n-1}(2P_{2n-2}+1)(2P_{2n}+1).\end{aligned}$$

Thus,

$$\frac{2P_{2n-2}}{P_{2n-1}(2P_{2n-2}+1)(2P_{2n}+1)}>\frac{1}{2P_{2mn}+1}.$$

It follows that

$$\sum_{k=n}^{mn}\frac{1}{P_{2k-1}}>\frac{1}{2P_{2n-2}+1}. \tag{3.15}$$

Again by elementary calculation and (2.1), for $k\geq 2$ we obtain

$$\begin{aligned}\frac{1}{2P_{2k-2}}-\frac{1}{P_{2k-1}}-\frac{1}{2P_{2k}}&=\frac{2P_{2k-1}\left(P_{2k}-P_{2k-2}\right)-4P_{2k-2}P_{2k}}{4P_{2k-2}P_{2k-1}P_{2k}}\\&=\frac{4P_{2k-1}^2-4P_{2k-2}P_{2k}}{4P_{2k-2}P_{2k-1}P_{2k}}\\&=\frac{1}{P_{2k-2}P_{2k-1}P_{2k}}\\&>0.\end{aligned}$$

Summing from n to mn gives

$$\frac{1}{2P_{2n-2}}-\sum_{k=n}^{mn}\frac{1}{P_{2k-1}}-\frac{1}{2P_{2mn}}>0.$$

Then

$$\begin{aligned}\sum_{k=n}^{mn} \frac{1}{P_{2k-1}} &< \frac{1}{2P_{2n-2}} - \frac{1}{2P_{2mn}} \\ &< \frac{1}{2P_{2n-2}}. \end{aligned} \tag{3.16}$$

Combining (3.15) and (3.16), the desired result (3.14) follows. Moreover, because of $2P_{2n-2} = P_{2n-1} - P_{2n-3}$, the identity (3.13) is immediate. □

Letting m tend to infinity in Theorem 3.5, we obtain

Corollary 3.6. *For any positive integer $n \geq 2$, we have*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{P_{2k-1}} \right)^{-1} \right\rfloor = 2P_{2n-2} = P_{2n-1} - P_{2n-3}. \tag{3.17}$$

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