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RAMSEY ORDERLY ALGEBRAS AS A NEW APPROACH TO RAMSEY ALGEBRAS

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Abstract

Ramsey algebras are algebras that induce Ramsey spaces, which are generalizations of the Ellentuck space and the Milliken space. Previous work has suggested a possible local version of Ramsey algebras induced by infinite sequences. We formulate this local version and call it Ramsey orderly algebra. In this paper, we present an introductory treatment of this new notion and provide justification for it to be a sound approach for further study in Ramsey algebras. The main connection is that an algebra is Ramsey if and only if each of its induced orderly algebra is Ramsey.

1 Introduction

Ramsey spaces as introduced by Carlson in [1] are generalizations of the Ellentuck space [3] and the Milliken space [8]. This notion of Carlson has since then attracted a considerable amount of interest due to its power to derive a proliferation of known standard Ramsey theoretic results based on the existence of certain Ramsey spaces of variable words [1]. Examples of standard results are the dual Ellentuck Theorem [2], the Graham-Rothschild theorem on n-parameter sets [4], and the Hales-Jewett Theorem [5]. A modern reference to the topic is [15].

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The topological property of a Ramsey space can, in fact, be reduced to some combinatorial property. This is essentially captured by the abstract version of Ellentuck's Theorem, first pointed out by Carlson in the same article. In the context of a Ramsey space associated to an algebra, such as a Ramsey space of variable words, that combinatorial property is reflected in the algebra itself. For this reason, Carlson conceived the notion of Ramsey algebras and its first results followed suit through the work of the first author [11–14].

Every semigroup is a Ramsey algebra. The author's work in [12] generalizes the result to any algebra having sufficiently many infinite sequences of the underlying set that induce what are here called orderly semigroups (see Example 3.5). It is this observation that motivated the introduction of orderly algebras (Definition 3.1). This notion shifts the subject of Ramsey algebra from a global perspective to a local, sequential one. Specifically, the characterization of Ramsey algebras can be reduced to the characterization of Ramsey orderly algebras.

This paper presents preliminary findings to justify the pertinence of this notion for the study of Ramsey algebras. The authors believe that approaching the subject of Ramsey algebras through orderly algebras might be better suited for further studies of the subject. Besides that, the paper aims to disseminate the subject of Ramsey algebras to the combinatorial and logical community to encourage participation in the study of this subject, which is at its infancy stage. One open problem in the subject is the characterization of Ramsey algebras in terms of the property of the underlying operations. At this point of time, little is known about which common algebras, apart from semigroups, are Ramsey.

2 Preliminary

The set of natural numbers and the set of positive integers are denoted by ω and $\mathbb{N}$ respectively. The set of infinite sequences in A is denoted by ${}^{\omega}A$. The cardinality of a set A is denoted by $|A|$.

To us an *algebra* is a pair $(A, \mathcal{F})$, where A is a nonempty set and $\mathcal{F}$ is a collection of operations on A, none of which is nullary. We will write $(A, \{f\})$ simply as (A, f).

We assume the reader is familiar with the syntax and semantics of first order logic. In particular, we are only concerned with purely functional first-order logic $\mathcal{L}$, that is, logic whose non-logical symbols are functional. Fix a list of (syntactic) variables, $v_0, v_1, v_2, \ldots$. The *index* of v_i is i. The *terms* of such a language are expressions built up in the usual way from the variables and the function symbols. An $\mathcal{L}$*-algebra* is an $\mathcal{L}$-structure in the usual sense.

Suppose that $\mathfrak{A}$ is an $\mathcal{L}$-algebra. An *assignment* is (identified with) an infinite sequence whose terms are elements of the *universe* $\|\mathfrak{A}\|$ of $\mathfrak{A}$. The *interpretation* of a term t under $\mathfrak{A}$ and assignment $\vec{a}$, denoted $t^{\mathfrak{A}}[\vec{a}]$, is defined inductively in the usual way.

For each algebra $\mathfrak{A} = (A, \mathcal{F})$, there is a natural language associated to it. Let $\mathcal{L}_{\mathcal{F}}$ denote the language $\{\, \underline{f} \mid f \in \mathcal{F} \,\}$ where $\underline{f}$ and f have the same arity for every $f \in \mathcal{F}$ and $\underline{f}$ and $\underline{g}$ are distinct whenever f and g are distinct. We will identify $\mathfrak{A}$ with the $\mathcal{L}_{\mathcal{F}}$-algebra whose universe is A and the interpretation of $\underline{f}$ is f for every $f \in \mathcal{F}$.

To discuss Ramsey algebras or Ramsey orderly algebras, the notion of a "term" is inadequate. We need a stronger notion.

Definition 2.1. Suppose that $\mathcal{L}$ is a language. An *orderly term* of $\mathcal{L}$ is a term of $\mathcal{L}$ such that the indices of the variables appearing in it from left to right is strictly increasing. The set of orderly terms of $\mathcal{L}$ is denoted by $\mathrm{OT}(\mathcal{L})$. If $t, t' \in \mathrm{OT}(\mathcal{L})$, then $t < t'$ means that the index of the last variable in t is less than the index of the first variable in t'. An infinite sequence $\vec{t}$ of orderly terms of $\mathcal{L}$ is *admissible* iff it is increasing with respect to $<$. The set of admissible sequences of $\mathcal{L}$ is denoted by $\mathrm{Adm}(\mathcal{L})$.

Definition 2.2. Suppose that $\mathfrak{A} = (A, \mathcal{F})$ is an algebra and $\vec{a}, \vec{b} \in {}^{\omega}A$. We say that $\vec{b}$ is a *reduction* of $\vec{a}$ (with respect to $\mathcal{F}$), denoted $\vec{b} \leq_{\mathcal{F}} \vec{a}$, iff there exists $\vec{t} \in \mathrm{Adm}(\mathcal{L}_{\mathcal{F}})$ such that $\vec{b}(i) = (\vec{t}(i))^{\mathfrak{A}}[\vec{a}]$ for all $i \in \omega$.

The relation $\leq_{\mathcal{F}}$ is reflexive and transitive.

Definition 2.3. Suppose that $\mathfrak{A} = (A, \mathcal{F})$ is an $\mathcal{L}$-algebra and $\vec{a} \in {}^{\omega}A$. The *set of finite reductions* of $\vec{a}$ (with respect to $\mathcal{F}$), denoted $\mathrm{FR}_{\mathcal{F}}(\vec{a})$, is defined by

$$\mathrm{FR}_{\mathcal{F}}(\vec{a}) = \{\, t^{\mathfrak{A}}[\vec{a}] \mid t \in \mathrm{OT}(\mathcal{L}) \,\}.$$

Definition 2.4. Suppose $(A, \mathcal{F})$ is an algebra. We say that $(A, \mathcal{F})$ is *Ramsey* iff for every $\vec{a} \in {}^{\omega}A$ and $X \subseteq A$, there exists $\vec{b} \leq_{\mathcal{F}} \vec{a}$ such that $\mathrm{FR}_{\mathcal{F}}(\vec{b})$ is either contained in or disjoint from X.

If for every $\vec{a} \in {}^{\omega}A$, there exists $\vec{b} \leq_{\mathcal{F}} \vec{a}$ such that $|\,\mathrm{FR}_{\mathcal{F}}(\vec{b})| = 1$, then $(A, \mathcal{F})$ is trivially Ramsey, and we say that it is a *degenerate Ramsey algebra*.

The following localized version of Ramsey algebra is very relevant to this work.

Definition 2.5. Suppose $(A, \mathcal{F})$ is an algebra and $\vec{a} \in {}^{\omega}A$. We say that $(A, \mathcal{F})$ is *Ramsey below* $\vec{a}$ iff for every $\vec{b} \leq_{\mathcal{F}} \vec{a}$ and $X \subseteq A$, there exists $\vec{c} \leq_{\mathcal{F}} \vec{b}$ such that $\mathrm{FR}_{\mathcal{F}}(\vec{c})$ is either contained in or disjoint from X.

Remark 2.6. *An algebra $(A, \mathcal{F})$ is Ramsey if and only if it is Ramsey below $\vec{a}$ for every $\vec{a} \in {}^{\omega}A$. Additionally, if $(A, \mathcal{F})$ is Ramsey below $\vec{a}$, then it is Ramsey below $\vec{b}$ whenever $\vec{b} \leq_{\mathcal{F}} \vec{a}$.*

We now address the intimate connection between Ramsey algebras and Ramsey spaces. The definition of a Ramsey space presented in our context here mirrors that given in [1]. A Ramsey space is called a topological Ramsey space in [15] and it has a slightly different axiomatization.

Definition 2.7. A *preorder with approximations* is a pair $\mathfrak{R} = (R, \leq)$ such that R is a nonempty set of infinite sequences and $\leq$ is a reflexive and transitive relation on R. For $n \in \omega$ and $\vec{a} \in R$, define

$$[n, \vec{a}] = \{\, \vec{b} \in R \mid \vec{b} \leq \vec{a} \text{ and } \langle \vec{b}(0), \ldots, \vec{b}(n-1) \rangle = \langle \vec{a}(0), \ldots, \vec{a}(n-1) \rangle \,\}.$$

The *natural topology* on $\mathfrak{R}$ is the topology generated by the sets $[n, \vec{a}]$.

Definition 2.8. Suppose $\mathfrak{R} = (R, \leq)$ is a preorder with approximations. Assume X is a subset of R. We say that X is *Ramsey* iff for every $n \in \omega$ and $\vec{a} \in R$, there exists $\vec{b} \in [n, \vec{a}]$ such that $[n, \vec{b}]$ is either contained in or disjoint from X. Assuming the Axiom of Choice, $\mathfrak{R}$ is a *Ramsey space* iff every subset of $\mathfrak{R}$ (endowed with the natural topology) having the property of Baire is Ramsey.

Our work concerns preorders with approximations associated to algebras.

Definition 2.9. Suppose $(A, \mathcal{F})$ is an algebra and $\vec{a} \in {}^{\omega}A$. Define $\mathrm{R}(A, \mathcal{F})$ to be the preorder with approximations $({}^{\omega}A, \leq_{\mathcal{F}})$ and $\mathrm{R}_{\vec{a}}(A, \mathcal{F})$ to be the preorder with approximations $(\{\, \vec{b} \in {}^{\omega}A \mid \vec{b} \leq_{\mathcal{F}} \vec{a} \,\}, \leq_{\mathcal{F}})$.

Theorem 2.10. *Suppose $\mathcal{F}$ is a finite collection of operations on a set A, none of which is unary, and $\vec{a} \in {}^{\omega}A$. Then*

1. $\mathrm{R}(A, \mathcal{F})$ *is a Ramsey space if and only if* $(A, \mathcal{F})$ *is a Ramsey algebra;*
2. $\mathrm{R}_{\vec{a}}(A, \mathcal{F})$ *is a Ramsey space if and only if* $(A, \mathcal{F})$ *is Ramsey below* $\vec{a}$*.*

Theorem 2.10*(1)* is a version of Lemma 4.14 in [1] using the notion of Ramsey algebra. In fact, $\mathrm{R}(A, \mathcal{F})$ was shown in [1] to satisfy the assumptions for the abstract version of Ellentuck's Theorem that play a key role. As Theorem 2.10*(2)* can be proved analogously, it is first stated in [13] without proof. Furthermore, Theorem 2.10*(1)* is strengthen in [13] to allow for the collection of operations to be appended by any collection of unary operations. As of this writing, the author claims that the same conclusion holds provided the underlying set of the algebra is countable. However, it remains unclear whether there exists a Ramsey algebra such that the corresponding space is not Ramsey.

Example 2.11. The empty algebra $(\omega, \emptyset)$ is Ramsey precisely due to the pigeonhole principle. Hence, $\mathrm{R}(\omega, \emptyset)$ is a Ramsey space. Identifying infinite subsets A and B of ω with strictly increasing sequences of natural numbers, we have $A \subseteq B$ if and only if $A \leq_{\emptyset} B$. Therefore, the Ellentuck space is actually isomorphic to the subspace of $\mathrm{R}(\omega, \emptyset)$ induced by the set $\{\, A \in {}^{\omega}\omega \mid A \text{ is strictly inceasing} \,\}$. This set is the basic open set $[0, \langle 0, 1, 2, \ldots \rangle]$ in the natural topology on $\mathrm{R}(\omega, \emptyset)$ and as such induces a Ramsey subspace.

Example 2.12. Suppose L is a finite alphabet and v is a distinct variable not contained in L. A *variable word over L* is a finite sequence w of elements of

$L \cup \{v\}$ such that the variable v occurs at least once in w. Denote the set of variable words over L by W. If $w \in W$ and $a \in L$, then $w(a)$ is the result of replacing every occurrence of v in w by a. The concatenation of two variable words w and w' is denoted by $w * w'$. Let $\mathcal{F}$ consist of $*$ and the following binary operations on W:

$$(w, w') \mapsto w * w'(a) \quad , \quad a \in L;$$

$$(w, w') \mapsto w(a) * w' \quad , \quad a \in L.$$

$\mathrm{R}(W, \mathcal{F})$ is a prototype of Ramsey spaces of variable words [1].

Hindman's Theorem [6] implies that $(\mathbb{N}, +)$ is a Ramsey algebra. In fact, the generalization in the next theorem is essentially due to Hindman's Theorem (see [7, Section 5.2]). Alternatively, it follows from the fact that the preorder with approximations associated to a semigroup is a Ramsey space (Theorem 6 in [1]).

Theorem 2.13. *Every semigroup is a Ramsey algebra.*

Remark 2.14. *Structures that satisfy the Moufang identities are very close to being semigroups. For example, under octonion multiplication, the nonzero octonions form a Moufang loop that is nonassociative. Nevertheless, this Moufang loop is not a Ramsey algebra (see [9]).*

If $\mathrm{R}(A, \mathcal{F})$ is a Ramsey space, then $(A, \mathcal{F})$ is trivially a Ramsey algebra because every subset of A induces a subset of ${}^{\omega}A$ that is clopen in the natural topology. In fact, the strength of being a Ramsey space bestows more Ramsey type property on the associated algebra, which is the reason behind the following analogue of the Milliken-Taylor Theorem [8, 10].

Theorem 2.15. *Suppose $\mathcal{F}$ is a finite collection of operations on a set A, none of which is unary, $\vec{a} \in {}^{\omega}A$, and $n \in \omega$. If $\mathfrak{A} = (A, \mathcal{F})$ is Ramsey below $\vec{a}$, then for every $\vec{b} \leq_{\mathcal{F}} \vec{a}$ and $X \subseteq A^n$, there exists $\vec{c} \leq_{\mathcal{F}} \vec{b}$ such that $[\mathrm{FR}_{\mathcal{F}}(\vec{c})]^n_<$ is either contained in or disjoint from X, where*

$$[\mathrm{FR}_{\mathcal{F}}(\vec{c})]^n_< = \{\, (t_1^{\mathfrak{A}}[\vec{c}], \ldots, t_n^{\mathfrak{A}}[\vec{c}]) \mid t_1, \ldots, t_n \in \mathrm{OT}(\mathcal{L}_{\mathcal{F}}) \text{ with } t_1 < \cdots < t_n \,\}.$$

Proof. Let $R = \{\, \vec{b} \in {}^{\omega}A \mid \vec{b} \leq_{\mathcal{F}} \vec{a} \,\}$. By Theorem 2.10, $\mathrm{R}_{\vec{a}}(A, \mathcal{F}) = (R, \leq_{\mathcal{F}})$ is a Ramsey space.

Fix $\vec{b} \leq_{\mathcal{F}} \vec{a}$ and $X \subseteq A^n$. Let

$$Y = \{\, \vec{d} \in R \mid (\vec{d}(0), \vec{d}(1), \ldots, \vec{d}(n-1)) \in X \,\}.$$

Clearly, $[n, \vec{d}] \subseteq Y$ for every $\vec{d} \in Y$. Hence, $Y = \bigcup_{\vec{d} \in Y} [n, \vec{d}]$ is open in the natural topology on $\mathrm{R}_{\vec{a}}(A, \mathcal{F})$. Thus Y has the property of Baire and so is Ramsey because $\mathrm{R}_{\vec{a}}(A, \mathcal{F})$ is Ramsey. Therefore, we can choose $\vec{c} \leq_{\mathcal{F}} \vec{b}$ such that $[0, \vec{c}] = \{\, \vec{d} \in R \mid \vec{d} \leq_{\mathcal{F}} \vec{c} \,\}$ is either contained in or disjoint from Y.

Suppose $(d_0, d_1, \ldots, d_{n-1}) \in [\mathrm{FR}_{\mathcal{F}}(\vec{c})]^n_<$ is arbitrary. By the definition of a reduction, $\langle d_0, d_1, \ldots, d_{n-1}\rangle$ can be extended easily to some reduction $\vec{d}$ of $\vec{c}$, meaning $\vec{d} \in [0, \vec{c}]$. Note that $\vec{d} \in Y$ if and only if $(d_0, d_1, \ldots, d_{n-1}) \in X$. It follows that $[\mathrm{FR}_{\mathcal{F}}(\vec{c})]^n_<$ is either contained in or disjoint from X. □

3 Orderly Algebras and Their Reductions

We now introduce the notion of orderly algebra. Its naturality and relevance to the study of Ramsey algebras will become clear by the time we reach Theorem 4.4.

Definition 3.1. Suppose $\mathcal{L}$ is a language. An *orderly $\mathcal{L}$-algebra* is a function A with domain $\mathrm{OT}(\mathcal{L})$ such that for each $f \in \mathcal{L}$ the following holds: if f is n-ary, then $\mathrm{A}(ft_1t_2\cdots t_n) = \mathrm{A}(ft'_1t'_2\cdots t'_n)$ whenever $t_1, t_2, \ldots, t_n, t'_1, t'_2, \ldots, t'_n$ are orderly terms of $\mathcal{L}$ with $t_1 < t_2 < \cdots < t_n$ and $t'_1 < t'_2 < \cdots < t'_n$ such that $\mathrm{A}(t_k) = \mathrm{A}(t'_k)$ for $1 \leq k \leq n$. The range of A, denoted $\|\mathrm{A}\|$, is called the *universe* of A.

Example 3.2. Suppose $\mathcal{L}$ is a language. If A is any constant function with domain $\mathrm{OT}(\mathcal{L})$, then A is what we call a *trivial orderly $\mathcal{L}$-algebra*.

Example 3.3. Suppose $\mathcal{L}$ is a language. Let $\mathrm{A}(t) = \{i \in \omega \mid v_i \text{ appears in } t\}$ for all $t \in \mathrm{OT}(\mathcal{L})$. Then A is an orderly $\mathcal{L}$-algebra.

Examples 3.2 and 3.3 are arbitrary in the sense that no particular algebra has played a role in defining the values of A in either case. Since we are interested in algebras, we would like to be able to obtain an orderly $\mathcal{L}$-algebra from a given algebra. This is what we do in the next definition.

Definition 3.4. Suppose $\mathcal{L}$ is a language, $\mathfrak{A}$ is an $\mathcal{L}$-algebra and $\vec{a} \in {}^{\omega}\|\mathfrak{A}\|$. The *orderly $\mathcal{L}$-algebra induced from $\mathfrak{A}$ by $\vec{a}$*, denoted $\mathfrak{A}_{\vec{a}}$, is defined by

$$\mathfrak{A}_{\vec{a}}(t) = t^{\mathfrak{A}}[\vec{a}] \text{ for all } t \in \mathrm{OT}(\mathcal{L}).$$

The fact that $\mathfrak{A}_{\vec{a}}$ is well-defined follows from the semantics of first order logic, namely $(ft_1t_2\cdots t_n)^{\mathfrak{A}}[\vec{a}] = f^{\mathfrak{A}}(t_1^{\mathfrak{A}}[\vec{a}], \ldots, t_n^{\mathfrak{A}}[\vec{a}])$ whenever $f \in \mathcal{L}$, say n-ary, and $t_1, \ldots, t_n$ are terms of $\mathcal{L}$.

Note that if $\mathfrak{A} = (A, \mathcal{F})$ is an $\mathcal{L}$-algebra and $\vec{a} \in {}^{\omega}A$, then $\|\mathfrak{A}_{\vec{a}}\| = \mathrm{FR}_{\mathcal{F}}(\vec{a})$.

Example 3.5. Suppose $\mathcal{L} = \{f\}$, where f is binary. An orderly $\mathcal{L}$-algebra A is an *orderly semigroup* iff $\mathrm{A}(fft_1t_2t_3) = \mathrm{A}(ft_1ft_2t_3)$ for every $t_1, t_2, t_3 \in \mathrm{OT}(\mathcal{L})$ with $t_1 < t_2 < t_3$. If A is an orderly semigroup, then in fact $\mathrm{A}(s) = \mathrm{A}(t)$ whenever the same variables occur in s and t (essentially Lemma 4.2 in [12]). Every orderly $\mathcal{L}$-algebra induced from a semigroup is an orderly semigroup.

It will be shown now that, conversely, every orderly $\mathcal{L}$-algebra is induced from some $\mathcal{L}$-algebra.

Suppose that $\mathcal{L}$ is a language and A is an orderly $\mathcal{L}$-algebra. For each fixed $f \in \mathcal{L}$, say f is n-ary, the map

$$(\mathrm{A}(t_1), \ldots, \mathrm{A}(t_n)) \mapsto \mathrm{A}(ft_1 \cdots t_n) \quad , \quad t_1 < t_2 < \cdots < t_n$$

is a well-defined n-ary partial operation on $\|\mathrm{A}\|$. Extend this map arbitrarily to an n-ary operation on $\|\mathrm{A}\|$, denoted by f^*.

Let $\mathfrak{A} = (\|\mathrm{A}\|, \{f^*\}_{f \in \mathcal{L}})$ and $\vec{a} = \langle \mathrm{A}(v_0), \mathrm{A}(v_1), \mathrm{A}(v_2), \cdots \rangle$. Then $\mathfrak{A}$ is canonically an $\mathcal{L}$-algebra with $f^{\mathfrak{A}} = f^*$ for each $f \in \mathcal{L}$. We claim that $\mathfrak{A}_{\vec{a}} = \mathrm{A}$. To see this, we argue by induction on the complexity of orderly terms that $\mathfrak{A}_{\vec{a}}(t) = \mathrm{A}(t)$ for all $t \in \mathrm{OT}(\mathcal{L})$. For each variable v_i, we have $\mathfrak{A}_{\vec{a}}(v_i) = v_i^{\mathfrak{A}}[\vec{a}] = \vec{a}(i) = \mathrm{A}(v_i)$. Now, assume $t = ft_1t_2 \cdots t_n$ for some n-ary $f \in \mathcal{L}$ and orderly terms $t_1 < t_2 < \cdots < t_n$. By the induction hypothesis, $\mathfrak{A}_{\vec{a}}(t_i) = t_i^{\mathfrak{A}}[\vec{a}] = \mathrm{A}(t_i)$ for each $1 \leq i \leq n$. By definition, $\mathfrak{A}_{\vec{a}}(t) = t^{\mathfrak{A}}[\vec{a}] = f^{\mathfrak{A}}(t_1^{\mathfrak{A}}[\vec{a}], t_2^{\mathfrak{A}}[\vec{a}], \ldots, t_n^{\mathfrak{A}}[\vec{a}]) = f^*(\mathrm{A}(t_1), \mathrm{A}(t_2), \ldots, \mathrm{A}(t_n)) = \mathrm{A}(ft_1t_2 \cdots t_n) = \mathrm{A}(t)$. Hence, we have proved the following theorem.

Theorem 3.6. *Suppose that $\mathcal{L}$ is a language and* A *is an orderly $\mathcal{L}$-algebra. Then there exists an $\mathcal{L}$-algebra $\mathfrak{A}$ with universe $\|\mathrm{A}\|$ such that* A *is induced from $\mathfrak{A}$.*

Definition 3.7. Suppose $\mathcal{L}$ is a language, $s \in \mathrm{OT}(\mathcal{L})$, and $\vec{t} \in \mathrm{Adm}(\mathcal{L})$. Define $s[\vec{t}]$ to be the orderly term of $\mathcal{L}$ obtained by replacing each variable v_i occurring in s by $\vec{t}(i)$.

Definition 3.8. Suppose $\mathcal{L}$ is a language and suppose A and B are orderly $\mathcal{L}$-algebras. We say that B is a *reduction* of A iff there exists $\vec{t} \in \mathrm{Adm}(\mathcal{L})$ such that for every $s \in \mathrm{OT}(\mathcal{L})$,

$$\mathrm{B}(s) = \mathrm{A}(s[\vec{t}]).$$

We say that B is a reduction of A *witnessed* by $\vec{t}$.

Remark 3.9.

1. *If* B *is a reduction of* A*, then* $\|\mathrm{B}\| \subseteq \|\mathrm{A}\|$.
2. *If* B *is a reduction of* A *and* C *is a reduction of* B*, then* C *is a reduction of* A.

The choice of the term "reduction" is justified by the following proposition.

Proposition 3.10. *Suppose $\mathcal{L}$ is a language, $\mathfrak{A} = (A, \mathcal{F})$ is an $\mathcal{L}$-algebra, and $\vec{a} \in {}^{\omega}A$. Then*

1. *$\vec{b} \leq_{\mathcal{F}} \vec{a}$ if and only if $\mathfrak{A}_{\vec{b}}$ is a reduction of $\mathfrak{A}_{\vec{a}}$ for each $\vec{b} \in {}^{\omega}A$;*
2. *if* B *is a reduction of $\mathfrak{A}_{\vec{a}}$ and $\vec{b}(i) = \mathrm{B}(v_i)$ for all $i \in \omega$, then $\vec{b}$ is a reduction of $\vec{a}$ and* $\mathrm{B} = \mathfrak{A}_{\vec{b}}$.

Proof. For the first part, fix $\vec{b} \in {}^{\omega}A$. Assume $\vec{b} \leq_{\mathcal{F}} \vec{a}$. Then choose $\vec{t} \in \mathrm{Adm}(\mathcal{L}_{\mathcal{F}})$ such that $\vec{b}(i) = (\vec{t}(i))^{\mathfrak{A}}[\vec{a}]$ for all $i \in \omega$. For every $s \in \mathrm{OT}(\mathcal{L}_{\mathcal{F}})$, by the substitution lemma, $\mathfrak{A}_{\vec{b}}(s) = s^{\mathfrak{A}}[\vec{b}] = (s[\vec{t}])^{\mathfrak{A}}[\vec{a}] = \mathfrak{A}_{\vec{a}}(s[\vec{t}])$. Thus $\mathfrak{A}_{\vec{b}}$ is a reduction of $\mathfrak{A}_{\vec{a}}$ witnessed by $\vec{t}$. Conversely, assume $\mathfrak{A}_{\vec{b}}$ is a reduction of $\mathfrak{A}_{\vec{a}}$ witnessed by some $\vec{t} \in \mathrm{Adm}(\mathcal{L}_{\mathcal{F}})$. Then $\vec{b}(i) = v_i^{\mathfrak{A}}[\vec{b}] = \mathfrak{A}_{\vec{b}}(v_i) = \mathfrak{A}_{\vec{a}}(\vec{t}(i)) = (\vec{t}(i))^{\mathfrak{A}}[\vec{a}]$ for all $i \in \omega$. Therefore, $\vec{b}$ is a reduction of $\vec{a}$.

For the second part, suppose B is a reduction of $\mathfrak{A}_{\vec{a}}$ witnessed by some $\vec{t} \in \mathrm{Adm}(\mathcal{L}_{\mathcal{F}})$ and $\vec{b}(i) = \mathrm{B}(v_i)$ for all $i \in \omega$. Then $\vec{b}(i) = \mathfrak{A}_{\vec{a}}(\vec{t}(i)) = (\vec{t}(i))^{\mathfrak{A}}[\vec{a}]$ for all $i \in \omega$, implying that $\vec{b}$ is a reduction of $\vec{a}$. Now, for every $s \in \mathrm{OT}(\mathcal{L}_{\mathcal{F}})$, $\mathrm{B}(s) = \mathfrak{A}_{\vec{a}}(s[\vec{t}]) = (s[\vec{t}])^{\mathfrak{A}}[\vec{a}] = s^{\mathfrak{A}}[\vec{b}] = \mathfrak{A}_{\vec{b}}(s)$. Therefore, $\mathrm{B} = \mathfrak{A}_{\vec{b}}$. □

Corollary 3.11. *Suppose $\mathcal{L}$ is a language and* A *is an orderly $\mathcal{L}$-algebra. If* B *and* C *are reductions of* A *such that* $\mathrm{B}(v_i) = \mathrm{C}(v_i)$ *for all* $i \in \omega$*, then* $\mathrm{B} = \mathrm{C}$.

Proof. By Theorem 3.6, $\mathrm{A} = \mathfrak{A}_{\vec{a}}$ for some $\mathcal{L}$-algebra $\mathfrak{A}$ and sequence $\vec{a} \in {}^{\omega}\|\mathrm{A}\|$. The conclusion then follows by Proposition 3.10*(2)*. □

The corollary states that the reduction of any given orderly $\mathcal{L}$-algebra is uniquely determined by its values on the variables.

4 Ramsey Orderly Algebras

This section is devoted to the connection between orderly $\mathcal{L}$-algebras and Ramsey algebras. Theorem 4.4 is the main result concerning the connection.

Definition 4.1. Suppose $\mathcal{L}$ is a language and A is an orderly $\mathcal{L}$-algebra. We say that A is *weakly Ramsey* iff for every $X \subseteq \|\mathrm{A}\|$, there exists a reduction B of A *homogeneous for X* in the sense that $\|\mathrm{B}\|$ is either contained in or disjoint from X. We say that A is *Ramsey* iff for every reduction B of A and $X \subseteq \|\mathrm{A}\|$, there exists a reduction C of B homogeneous for X.

Remark 4.2. *1. If* A *is Ramsey, then it is weakly Ramsey.*

2. If some reduction of A *is weakly Ramsey, then* A *is weakly Ramsey.*

3. If A *is Ramsey, then every reduction of* A *is Ramsey.*

Proposition 4.3. *Suppose that $\mathfrak{A} = (A, \mathcal{F})$ is an $\mathcal{L}$-algebra and $\vec{a} \in {}^{\omega}A$. Then $\mathfrak{A}_{\vec{a}}$ is a Ramsey orderly $\mathcal{L}$-algebra if and only if $\mathfrak{A}$ is Ramsey below $\vec{a}$.*

Proof. Proposition 3.10 will be applied repeatedly. Assume $\mathfrak{A}_{\vec{a}}$ is Ramsey. Suppose $\vec{b} \leq_{\mathcal{F}} \vec{a}$ and $X \subseteq A$. Then $\mathfrak{A}_{\vec{b}}$ is a reduction of $\mathfrak{A}_{\vec{a}}$. Choose a reduction C of $\mathfrak{A}_{\vec{b}}$ homogeneous for $X \cap \|\mathfrak{A}_{\vec{a}}\|$. We must have $\mathrm{C} = \mathfrak{A}_{\vec{c}}$ for some $\vec{c} \leq_{\mathcal{F}} \vec{b}$. By Remark 3.9, $\|\mathrm{C}\| \subseteq \|\mathfrak{A}_{\vec{a}}\|$. Together with $\|\mathrm{C}\| = \mathrm{FR}_{\mathcal{F}}(\vec{c})$, it follows that $\mathrm{FR}_{\mathcal{F}}(\vec{c})$ is either contained in or disjoint from X. Therefore, $\mathfrak{A}$ is Ramsey below $\vec{a}$.

Conversely, assume $\mathfrak{A}$ is Ramsey below $\vec{a}$. Suppose $X \subseteq \|\mathfrak{A}_{\vec{a}}\|$ and B is a reduction of $\mathfrak{A}_{\vec{a}}$, say $\mathrm{B} = \mathfrak{A}_{\vec{b}}$ for some $\vec{b} \leq_{\mathcal{F}} \vec{a}$. Choose $\vec{c} \leq_{\mathcal{F}} \vec{b}$ such that $\mathrm{FR}_{\mathcal{F}}(\vec{c})$ is either contained in or disjoint from X. Then $\mathfrak{A}_{\vec{c}}$ is a reduction of B homogeneous for X because $\|\mathfrak{A}_{\vec{c}}\| = \mathrm{FR}_{\mathcal{F}}(\vec{c})$. □

Theorem 4.4. *Suppose that $\mathfrak{A} = (A, \mathcal{F})$ is an $\mathcal{L}$-algebra. The following are equivalent.*

1. *$\mathfrak{A}$ is a Ramsey algebra.*
2. *$\mathfrak{A}_{\vec{a}}$ is a Ramsey orderly $\mathcal{L}$-algebra for all $\vec{a} \in {}^{\omega}A$.*
3. *$\mathfrak{A}_{\vec{a}}$ is a weakly Ramsey orderly $\mathcal{L}$-algebra for all $\vec{a} \in {}^{\omega}A$.*

Proof. The equivalence of *(1)* and *(2)* follows from Remark 2.6 and Theorem 4.3 while *(2)* immediately implies *(3)*. Assume *(3)* holds. Fix $\vec{a} \in {}^{\omega}A$. Suppose $X \subseteq \|\mathfrak{A}_{\vec{a}}\|$ and B is a reduction of $\mathfrak{A}_{\vec{a}}$. By Proposition 3.10*(2)* and our assumption, B is weakly Ramsey. Choose a reduction C of B homogeneous for $X \cap \|\mathrm{B}\|$. This C is also homogeneous for X as required. □

For the rest of this section, we present an assorted array of elementary results. We begin with showing that if an orderly $\mathcal{L}$-algebra is a one-to-one function, then it cannot be Ramsey.

Theorem 4.5. *Suppose that $\mathcal{L}$ is a nonempty language and* A *is an orderly $\mathcal{L}$-algebra. If* A *is one-to-one, then it is not weakly Ramsey and thus not Ramsey.*

Proof. We define a subset X of $\|\mathrm{A}\|$ as follows. Fix a function symbol $f \in \mathcal{L}$ and say f is n-ary. For every $t \in \mathrm{OT}(\mathcal{L})$, let $\mathrm{A}(t) \in X$ iff the number of f occurring before the first variable in t is even. Since A is one-to-one, the set X is well-defined. Suppose B is any reduction of A, say witnessed by $\langle t_0, t_1, t_2, \dots \rangle$. By definition, $\mathrm{B}(v_0) = \mathrm{A}(t_0)$ and $\mathrm{B}(fv_0v_1 \cdots v_n) = \mathrm{A}(ft_0t_1 \cdots t_n)$. Obviously, $\mathrm{A}(t_0) \in X$ if and only if $\mathrm{A}(ft_0t_1 \cdots t_n) \notin X$. Hence, B is not homogeneous for X. Therefore, A is not weakly Ramsey. □

Corollary 4.6. *If f is a one-to-one binary operation on an infinite set A, then (A, f) is not a Ramsey algebra.*

Sketch of proof. Through careful analysis, a sequence $\vec{a}$ such that the induced orderly algebra $\mathfrak{A}_{\vec{a}}$ is one-to-one can be constructed inductively. Hence, by Theorems 4.4 and 4.5, (A, f) is not a Ramsey algebra. □

Theorem 4.7. *Suppose that $\mathcal{L}$ is a language and* A *is an orderly $\mathcal{L}$-algebra with a finite universe. Then* A *is weakly Ramsey if and only if* A *has a reduction that is a trivial orderly $\mathcal{L}$-algebra.*

Proof. The backward direction is immediate. Now, suppose A is weakly Ramsey. Assume A has no reduction that is trivial. For this proof, let us say that a reduction B of A is minimal iff no reduction of B has a universe with smaller cardinality.

Claim. *Suppose* B *is a minimal reduction of* A *and* $b \in \|\mathrm{B}\|$. *Then there exists a reduction* C *of* B *such that* $\mathrm{C}(v_i) = b$ *for every* $i \in \omega$.

Proof of claim. Such C exists if there are orderly terms $t_0 < t_1 < t_2 < \cdots$ such that $\mathrm{B}(t_i) = b$ for all $i \in \omega$. It suffices to show that for every natural number n, there exists $t \in \mathrm{OT}(\mathcal{L})$ such that $\mathrm{B}(t) = b$ and the index of the first variable of t is greater than n. Assume this is not the case. Then the first variable of any $t \in \mathrm{OT}(\mathcal{L})$ such that $\mathrm{B}(t) = b$ is bounded, say by N. Let C be the reduction of B witnessed by $v_{N+1} < v_{N+2} < v_{N+3} < \cdots$. Clearly, $b \notin \|\mathrm{C}\|$ and hence $\|\mathrm{C}\|$ is a proper subset of $\|\mathrm{B}\|$, contradicting the minimality of B. □

Claim. *If* B *and* C *are minimal reductions of* A, *then* $\|\mathrm{B}\|$ *and* $\|\mathrm{C}\|$ *are either equal or disjoint.*

Proof of claim. We argue by contradiction. Assume $b \in \|\mathrm{B}\| \cap \|\mathrm{C}\|$. Then by the previous claim, there is a reduction D of B such that $\mathrm{D}(v_i) = b$ for all $i \in \omega$. Likewise, there is a reduction E of C such that $\mathrm{E}(v_i) = b$ for all $i \in \omega$. Since D and E are both reductions of A, by Corollary 3.11, D = E. It follows that $\|\mathrm{B}\| = \|\mathrm{D}\| = \|\mathrm{E}\| = \|\mathrm{C}\|$. □

By the assumption, the universe of any minimal reduction of A has size at least two. Let X consist of exactly one representative from the universe of each minimal reduction of A such that if two minimal reductions have the same universe, they share the same representative. By our claim, $\|\mathrm{A}\| \setminus X$ contains at least some element from the universe of each minimal reduction of A. Since A is weakly Ramsey, choose a reduction B of A homogeneous for X. Such B can be further required to be a minimal reduction of A. However, this contradicts our choice of X. □

The following corollary of Theorem 4.7 has a direct proof in [13].

Corollary 4.8. *Every finite Ramsey algebra is a degenerate Ramsey algebra.*

The next theorem is a reformulation of Corollary 4.9 in [12] into the context of orderly algebra. The proof of the nontrivial backward direction is a minor modification of the proof of Theorem 4.8 in [12] and is thus omitted.

Theorem 4.9. *Suppose* $\mathcal{L}$ *is a language that contains some binary function symbol and* A *is an orderly* $\mathcal{L}$*-algebra. Then* A *is weakly Ramsey if and only if for every* $X \subseteq \|\mathrm{A}\|$, *there exists a reduction* B *of* A pre-homogeneous for X, *in the sense that for every* $t_1, t_2 \in \mathrm{OT}(\mathcal{L})$ *such that the same variables appear in both orderly terms,* $\mathrm{B}(t_1) \in X$ *if and only if* $\mathrm{B}(t_2) \in X$.

Remark 4.10. *The assumption that $\mathcal{L}$ contains some binary function symbol is necessary. Otherwise, let $\mathfrak{A} = (\mathbb{N}, +_3)$, where $+_3(x,y,z) = x+y+z$ for all $x,y,z \in \mathbb{N}$. If $\vec{a}$ is any infinite sequence of odd numbers, for example, then $\mathfrak{A}_{\vec{a}}$ is not weakly Ramsey although $\mathfrak{A}_{\vec{a}}$ is trivially pre-homogeneous for any subset of $\|\mathfrak{A}_{\vec{a}}\|$.*

Corollary 4.11. *Every orderly semigroup is Ramsey.*

Finally, we present the analogue of Theorem 2.15 for orderly algebras.

Theorem 4.12. *Suppose $\mathcal{L}$ is a finite language without unary function symbol and* A *is an orderly $\mathcal{L}$-algebra. If* A *is Ramsey, then for every reduction* B *of* A*, $n \in \omega$, and $X \subseteq \|\mathrm{A}\|^n$, there exists a reduction* C *of* B *such that $\|\mathrm{C}\|^n_<$ is either contained in or disjoint from X, where*

$$\|\mathrm{C}\|^n_< = \{\, (\mathrm{C}(t_1), \dots, \mathrm{C}(t_n)) \mid t_1, \dots, t_n \in \mathrm{OT}(\mathcal{L}) \text{ with } t_1 < \cdots < t_n \,\}.$$

Proof. Suppose B is a reduction of A, $n \in \omega$, and $X \subseteq \|\mathrm{A}\|^n$. By Theorem 3.6, $\mathrm{A} = \mathfrak{A}_{\vec{a}}$ for some $\mathcal{L}$-algebra $\mathfrak{A} = (\|\mathrm{A}\|, \mathcal{F})$ and $\vec{a} \in {}^{\omega}\|\mathrm{A}\|$. Hence, by Proposition 3.10, $\mathrm{B} = \mathfrak{A}_{\vec{b}}$ for some $\vec{b} \leq_{\mathcal{F}} \vec{a}$. Since A is Ramsey, by Theorem 4.3, $\mathfrak{A}$ is Ramsey below $\vec{a}$. By Theorem 2.15, there exists $\vec{c} \leq_{\mathcal{F}} \vec{b}$ such that $[\mathrm{FR}_{\mathcal{F}}(\vec{c})]^n_<$ is either contained in or disjoint from X. Then $\mathfrak{A}_{\vec{c}}$ is a reduction of B and it remains to note that $\|\mathfrak{A}_{\vec{c}}\|^n_< = [\mathrm{FR}_{\mathcal{F}}(\vec{c})]^n_<$. □

5 A Case Study

In addition to applying the notion of orderly $\mathcal{L}$-algebras to the study of Ramsey algebras, the case study in this section is intended to demonstrate how the notion in question facilitates the study of Ramsey algebra.

Throughout this section, suppose $\mathcal{L} = \{f\}$, where f is binary, and fix an orderly $\mathcal{L}$-algebra A. We will define an orderly $\mathcal{L}$-algebra, denoted $\sharp$A, with universe a subset of $\|\mathrm{A}\| \times \|\mathrm{A}\|$ and show that $\sharp$A is Ramsey provided that A is Ramsey. To do this, every orderly term t of $\mathcal{L}$ is associated with a pair of (orderly) terms of $\mathcal{L}$, denoted (t^x, t^y), defined inductively in the following way:

$$(v_i^x, v_i^y) = (v_{2i}, v_{2i+1}) \text{ for all } i \in \omega;$$

$$((fst)^x, (fst)^y) = (fs^x s^y, ft^x t^y) \text{ for all } s, t \in \mathrm{OT}(\mathcal{L}) \text{ with } s < t.$$

Claim. *$t^x, t^y \in \mathrm{OT}(\mathcal{L})$ and $t^x < t^y$ for every $t \in \mathrm{OT}(\mathcal{L})$.*

Proof. In fact, v_i occurs in t if and only if each of v_{2i} and v_{2i+1} occurs in either t^x or t^y. It is straightforward to prove this stronger claim by induction on the complexity of orderly terms. □

Now, for every $t \in \mathrm{OT}(\mathcal{L})$, define $\sharp\mathrm{A}(t) = (\mathrm{A}(t^x), \mathrm{A}(t^y))$.

Claim. $\sharp\mathrm{A}$ *is an orderly* $\mathcal{L}$*-algebra and* $\|\sharp\mathrm{A}\| \subseteq \|\mathrm{A}\|^2_<$.

Proof. Suppose $s_1 < t_1$, $s_2 < t_2$, $\sharp\mathrm{A}(s_1) = \sharp\mathrm{A}(s_2)$, and $\sharp\mathrm{A}(t_1) = \sharp\mathrm{A}(t_2)$. We need to show that $\sharp\mathrm{A}(fs_1t_1) = \sharp\mathrm{A}(fs_2t_2)$. From $\sharp\mathrm{A}(s_1) = \sharp\mathrm{A}(s_2)$, it follows that $\mathrm{A}(s_1^x) = \mathrm{A}(s_2^x)$ and $\mathrm{A}(s_1^y) = \mathrm{A}(s_2^y)$. Since A is an orderly L-algebra, $\mathrm{A}(fs_1^x s_1^y) = \mathrm{A}(fs_2^x s_2^y)$. Similarly, $\mathrm{A}(ft_1^x t_1^y) = \mathrm{A}(ft_2^x t_2^y)$. Therefore, $\sharp\mathrm{A}(fs_1t_1) = (\mathrm{A}(fs_1^x s_1^y), \mathrm{A}(ft_1^x t_1^y)) = (\mathrm{A}(fs_2^x s_2^y), \mathrm{A}(ft_2^x t_2^y)) = \sharp\mathrm{A}(fs_2t_2)$ as required. The second part is immediate. ☐

Claim. *Suppose* B *is a reduction of* $\sharp\mathrm{A}$ *witnessed by* $\vec{t} = \langle t_0, t_1, t_2, \dots\rangle$*. Let* C *be the reduction of* A *witnessed by* $\vec{t'} = \langle ft_0^x t_0^y, ft_1^x t_1^y, ft_2^x t_2^y, \dots\rangle$*. If* D *is a reduction of* C,[1] *then* $\sharp\mathrm{D}$ *is a reduction of* B.

Proof. First of all, the following substitution property can be proved by induction on the complexity of orderly terms:

$$s[\vec{t'}] = f(s[\vec{t}])^x(s[\vec{t}])^y \text{ for all } s \in \mathrm{OT}(\mathcal{L}).$$

For the base step, $v_i[\vec{t'}] = ft_i^x t_i^y = f(v_i[\vec{t}])^x(v_i[\vec{t}])^y$. For the induction step,

$$\begin{aligned}(fss')[\vec{t'}] = fs[\vec{t'}]s'[\vec{t'}] = ff(s[\vec{t}])^x(s[\vec{t}])^y f(s'[\vec{t}])^x(s'[\vec{t}])^y = \\ f\big(fs[\vec{t}]s'[\vec{t}]\big)^x\big(fs[\vec{t}]s'[\vec{t}]\big)^y = f\big((fss')[\vec{t}]\big)^x\big((fss')[\vec{t}]\big)^y.\end{aligned}$$

Suppose D is a reduction of C witnessed by $\vec{u} = \langle u_0, u_1, u_2, \dots\rangle$. Let $\vec{u'} = \langle fu_0u_1, fu_2u_3, fu_4u_5, \dots\rangle$. Again, it can be proved similarly by induction on the complexity of orderly terms that

$$s[\vec{u'}] = (fs^x s^y)[\vec{u}] \text{ for all } s \in \mathrm{OT}(\mathcal{L}).$$

Now, we will show that $\sharp\mathrm{D}$ is a reduction of B witnessed by $\vec{u'}$. Fix $s \in \mathrm{OT}(\mathcal{L})$. Then by definition,

$$\sharp\mathrm{D}(s) = (\mathrm{D}(s^x), \mathrm{D}(s^y)) = (\mathrm{C}(s^x[\vec{u}]), \mathrm{C}(s^y[\vec{u}])) = (\mathrm{A}(s^x[\vec{u}][\vec{t'}]), \mathrm{A}(s^y[\vec{u}][\vec{t'}])).$$

By the first substitution property,

$$s^x[\vec{u}][\vec{t'}] = f(s^x[\vec{u}][\vec{t}])^x(s^x[\vec{u}][\vec{t}])^y = \big(f(s^x[\vec{u}][\vec{t}])(s^y[\vec{u}][\vec{t}])\big)^x = \big((fs^x s^y)[\vec{u}][\vec{t}]\big)^x.$$

Similarly, $s^y[\vec{u}][\vec{t'}] = \big((fs^x s^y)[\vec{u}][\vec{t}]\big)^y$. Therefore, $\sharp\mathrm{D}(s) = \sharp\mathrm{A}\big((fs^x s^y)[\vec{u}][\vec{t}]\big) = \mathrm{B}\big((fs^x s^y)[\vec{u}]\big) = \mathrm{B}(s[\vec{u'}])$ as required. ☐

Theorem 5.1. *If* A *is Ramsey, then* $\sharp\mathrm{A}$ *is Ramsey.*

[1] This requirement on D is essential. For example, say $\mathrm{B} = \sharp\mathrm{A}$. If D is a reduction of A witnessed by $v_0 < v_2 < v_4 < \cdots$, then $\sharp\mathrm{D}$ need not be a reduction of B.

Proof. Suppose B is a reduction of $\sharp$A witnessed by $\vec{t} = \langle t_0, t_1, t_2, \dots \rangle$ and $X \subseteq \|\sharp \mathrm{A}\|$. Let C be the reduction of A witnessed by $\vec{t'} = \langle f t_0^x t_0^y, f t_1^x t_1^y, f t_2^x t_2^y, \dots \rangle$. Since A is Ramsey and $X \subseteq \|\mathrm{A}\|^2$, by Theorem 4.12, choose a reduction D of C such that $\|\mathrm{D}\|^2_<$ is either contained in disjoint from X. By the previous claim, $\sharp$D is a reduction of B. Finally, since $\|\sharp \mathrm{D}\| \subseteq \|\mathrm{D}\|^2_<$, it follows that $\sharp$D is homogeneous for X. $\square$

Theorem 5.2. *Suppose g is a binary operation on a set A and h is an operation on A^2 defined by*

$$h((x_1, y_1), (x_2, y_2)) = (g(x_1, y_1), g(x_2, y_2)).$$

If (A, g) is a Ramsey algebra, then (A^2, h) is also a Ramsey algebra.

Proof. Let $\mathfrak{A}$ and $\mathfrak{B}$ denote the $\mathcal{L}$-algebras (A, g) and (A^2, h) respectively. Suppose $\vec{b} = \langle (x_i, y_i) \rangle_{i \in \omega} \in {}^{\omega}(A^2)$. Let $\vec{a} = \langle x_0, y_0, x_1, y_1, x_2, y_2, \dots \rangle$. By Theorems 4.4 and 5.1, it suffices to show that $\mathfrak{B}_{\vec{b}} = \sharp \mathfrak{A}_{\vec{a}}$. We prove this by induction on the complexity of orderly terms. For the base case, $\mathfrak{B}_{\vec{b}}(v_i) = (x_i, y_i) = (\mathfrak{A}_{\vec{a}}(v_{2i}), \mathfrak{A}_{\vec{a}}(v_{2i+1})) = \sharp \mathfrak{A}_{\vec{a}}(v_i)$. For the induction step, consider $fst \in \mathrm{OT}(\mathcal{L})$. By the induction hypothesis, $s^{\mathfrak{B}}[\vec{b}] = \mathfrak{B}_{\vec{b}}[s] = \sharp \mathfrak{A}_{\vec{a}}(s) = \big(\mathfrak{A}_{\vec{a}}(s^x), \mathfrak{A}_{\vec{a}}(s^y)\big) = \big((s^x)^{\mathfrak{A}}[\vec{a}], (s^y)^{\mathfrak{A}}[\vec{a}]\big)$. Similarly, we have $t^{\mathfrak{B}}[\vec{b}] = \big((t^x)^{\mathfrak{A}}[\vec{a}], (t^y)^{\mathfrak{A}}[\vec{a}]\big)$. Therefore, $\mathfrak{B}_{\vec{b}}(fst) = f^{\mathfrak{B}}(s^{\mathfrak{B}}[\vec{b}], t^{\mathfrak{B}}[\vec{b}]) = h\Big(\big((s^x)^{\mathfrak{A}}[\vec{a}], (s^y)^{\mathfrak{A}}[\vec{a}]\big) , \big((t^x)^{\mathfrak{A}}[\vec{a}], (t^y)^{\mathfrak{A}}[\vec{a}]\big) \Big)$. By the definition of h, this equals $\Big(g\big((s^x)^{\mathfrak{A}}[\vec{a}], (s^y)^{\mathfrak{A}}[\vec{a}]\big) , g\big((s^x)^{\mathfrak{A}}[\vec{a}], (s^y)^{\mathfrak{A}}[\vec{a}]\big) \Big) = \big((fs^x s^y)^{\mathfrak{A}}[\vec{a}], (ft^x t^y)^{\mathfrak{A}}[\vec{a}]\big) = (\mathfrak{A}_{\vec{a}}(fs^x s^y), \mathfrak{A}_{\vec{a}}(ft^x t^y)) = \sharp \mathfrak{A}_{\vec{a}}(fst)$. $\square$

Corollary 5.3. *There exists a Ramsey algebra (A, f) where f is a nowhere associative binary operation, meaning $f(f(a, b), c) \neq f(a, f(b, c))$ for every $(a, b, c) \in A^3$.*

Proof. Let $g((x_1, y_1), (x_2, y_2)) = (x_1 + y_1, x_2 + y_2)$ for all $(x_1, y_1), (x_2, y_2) \in \mathbb{N}^2$. It is easy to verify that g is nowhere associative. Since $(\mathbb{N}, +)$ is a Ramsey algebra for being a semigroup, by Theorem 5.2, $(\mathbb{N}^2, g)$ is a Ramsey algebra. $\square$

The results in this section can be extended analogously to the n-ary case.

6 Concluding Remarks

The paper begins with a brief historical account of the subject, followed by a formal introduction to the notion of a Ramsey algebra and its connection with Ramsey spaces.

In an attempt to facilitate the study of the characterization of Ramsey algebras, we have introduced the notion of an orderly $\mathcal{L}$-algebra based on the observation that whether or not an algebra is a Ramsey algebra can be cast in terms of infinite sequences in the underlying set of the algebra. This observation culminates in the notion of an orderly semigroup, an orderly algebra every semigroup would induce. Such a sufficient condition for an orderly algebra to be Ramsey is contrasted with the fact that even algebras closely resembling semigroups can fail to be Ramsey (cf. [9]), further suggesting that the crucial aspect that determines if an algebra is Ramsey lies in sequences and their reduction properties.

We ended our paper with a case study to demonstrate the facility afforded by orderly algebras in the study of Ramsey algebra. We conclude the paper with an invitation to the readers to pursue the study of Ramsey algebras towards a their characterization, particularly to apply the notion of Ramsey orderly algebras in such a study.

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