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EXISTENCE RESULTS FOR NAVIER PROBLEMS WITH DEGENERATED (p, q)-LAPLACIAN AND (p, q)-BIHARMONIC OPERATORS IN WEIGHTED SOBOLEV SPACES

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Abstract

In this article, we prove the existence and uniqueness of solutions for the Navier problem

$$(P)\begin{cases}\Delta\big[\omega(x)\,|\Delta u|^{p-2}\Delta u+\nu(x)\,|\Delta u|^{q-2}\Delta u\big]-\operatorname{div}\big[\omega(x)\,|\nabla u|^{p-2}\nabla u+\nu(x)\,|\nabla u|^{q-2}\nabla u\big]\\ =f(x)-\operatorname{div}(G(x)),\quad\text{in}\ \ \Omega,\\ u(x)=\Delta u=0,\quad\text{in}\ \ \partial\Omega,\end{cases}$$

where Ω is a bounded open set of $\mathbb{R}^N$ $(N\geq 2)$, $\dfrac{f}{\omega}\in L^{p'}(\Omega,\omega)$ and $\dfrac{G}{\nu}\in[L^{q'}(\Omega,\nu)]^N$.

1 Introduction

The main purpose of this paper (see Theorem 3.2) is to establish the existence and uniqueness of solutions for the Navier problem

$$(P)\begin{cases}Lu(x)=f(x)-\operatorname{div}(G(x)),\quad\text{in}\ \ \Omega,\\ u(x)=\Delta u(x)=0,\quad\text{in}\ \ \partial\Omega,\end{cases}$$

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where

$$Lu(x) = \Delta\big[\omega(x)\,|\Delta u|^{p-2}\Delta u + \nu(x)\,|\Delta u|^{q-2}\Delta u\big] - \mathrm{div}\big[\omega(x)|\nabla u|^{p-2}\nabla u + \nu(x)\,|\nabla u|^{q-2}\nabla u)\big],$$

$\Omega\subset\mathbb{R}^N$ is a bounded open set, $\dfrac{f}{\omega}\in L^{p'}(\Omega,\omega)$, $\dfrac{G}{\nu}\in[L^{q'}(\Omega,\nu)]^N$, ω and ν are two weight functions (i.e., ω and ν are locally integrable functions on $\mathbb{R}^N$ such that $0 < \omega(x) < \infty$ and $0 < \nu(x) < \infty$ a.e. $x\in\mathbb{R}^N$), Δ is the Laplacian operator, $1 < q < p < \infty$, $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$.

For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [1], [4], [5], [7], [8] and [11]). The type of a weight depends on the equation type.

A class of weights, which is particularly well understood, is the class of A_p weights that was introduced by B.Muckenhoupt in the early 1970's (see [8]). These classes have found many useful applications in harmonic analysis (see [9] and 10]). Another reason for studying A_p-weights is the fact that powers of the distance to submanifolds of $\mathbb{R}^N$ often belong to A_p (see [3] and [11]). There are, in fact, many interesting examples of weights (see [7] for p-admissible weights).

In the non-degenerate case (i.e. with $\omega(x) \equiv 1$), for all $f\in L^p(\Omega)$ the Poisson equation associated with the Dirichlet problem

$$\begin{cases} -\Delta u = f(x), \text{ in } \Omega \\ u(x) = 0, \text{ in } \partial\Omega \end{cases}$$

is uniquely solvable in $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ (see [6]), and the nonlinear Dirichlet problem

$$\begin{cases} -\Delta_p u = f(x), \text{ in } \Omega \\ u(x) = 0, \text{ in } \partial\Omega \end{cases}$$

is uniquely solvable in $W_0^{1,p}(\Omega)$ (see [2]), where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator. In the degenerate case, the degenerated p-Laplacian has been studied in [3].

The paper is organized as follow. In Section 2 we present the definitions and basic results. In Section 3 we prove our main result about existence and uniqueness of solutions for problem (P).

2 Definitions and basic results

Let Ω be an open set in $\mathbb{R}^n$. By the symbol $\mathcal{W}(\Omega)$ we denote the set of all measurable a.e. in Ω positive and finite functions $\omega = \omega(x)$, $x\in\Omega$. Elements of $\mathcal{W}(\Omega)$ will be called weight functions. Every weight ω gives rise to a measure on the measurable subsets of $\mathbb{R}^N$ through integration. This measure will be denoted by μ_ω. Thus, $\mu_\omega(E) = \int_E \omega(x)\,dx$ for measurable sets $E \subset \mathbb{R}^N$.

Definition 2.1. Let $1 \leq p < \infty$. A weight ω is said to be an A_p-weight, if there is a positive constant C such that, for every ball $B \subset \mathbb{R}^N$

$$\left(\frac{1}{|B|}\int_B \omega(x)\,dx\right)\left(\frac{1}{|B|}\int_B \omega^{1/(1-p)}(x)\,dx\right)^{p-1} \leq C, \quad \text{if} \quad p > 1,$$
$$\left(\frac{1}{|B|}\int_B \omega(x)\,dx\right)\left(\operatorname*{ess\,sup}_{x \in B} \frac{1}{\omega(x)}\right) \leq C, \quad \text{if} \quad p = 1,$$

where $|.|$ denotes the N-dimensional Lebesgue measure in $\mathbb{R}^N$. The infimum over all such constants C is called the A_p - constant of ω and is dnotaded by $C_{p,\omega}$.

If $1 < q \leq p$, then $A_q \subset A_p$ (see [5], [7] or [11] for more information about A_p-weights). As an example of an A_p-weight, the function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^N$, is in A_p if and only if $-N < \alpha < N(p-1)$ (see [11], Chapter IX, Corollary 4.4). If $\varphi \in BMO(\mathbb{R}^N)$, then $\omega(x) = \mathrm{e}^{\alpha\,\varphi(x)} \in A_2$ for some $\alpha > 0$ (see [9]).

Remark 2.1. If $\omega \in A_p$, $1 < p < \infty$, then

$$\left(\frac{|E|}{|B|}\right)^p \leq C_{p,\omega}\,\frac{\mu_\omega(E)}{\mu_\omega(B)}$$

for all measurable subsets E of B (see *15.5 strong doubling property* in [7]). Therefore, $\mu_\omega(E) = 0$ if and only if $|E| = 0$; so there is no need to specify the measure when using the ubiquitous expression almost everywhere and almost every, both abbreviated a.e..

Definition 2.2. Let $\Omega \subset \mathbb{R}^n$ a bounded open set, $\omega \in \mathcal{W}(\Omega)$ and $1 \leq p < \infty$. We shall denote by $L^p(\Omega, \omega)$ the Banach space of all measurable functions f defined in Ω for which

$$\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f(x)|^p \omega(x)\,dx\right)^{1/p} < \infty.$$

We denote $[L^p(\Omega,\omega)]^N = L^p(\Omega,\omega) \times \ldots \times L^p(\Omega,\omega)$.

Remark 2.2. If $\omega \in A_p$, $1 < p < \infty$, then since $\omega^{-1/(p-1)}$ is locally integrable, we have $L^p(\Omega,\omega) \subset L^1_{\text{loc}}(\Omega)$ (see [11], Remark 1.2.4). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$.

Definition 2.3. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $1 < p < \infty$, k be a non-negative integer and $\omega \in A_p$. We shall denote by $W^{k,p}(\Omega,\omega)$, the weighted

Sobolev spaces, the set of all functions $u \in L^p(\Omega,\omega)$ with weak derivatives $D^\alpha u \in L^p(\Omega,\omega)$, $1 \le |\alpha| \le k$. The norm in the space $W^{k,p}(\Omega,\omega)$ is defined by

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \left(\int_\Omega |u(x)|^p \omega(x)\, dx + \sum_{1 \le |\alpha| \le k} \int_\Omega |D^\alpha u(x)|^p \omega(x)\, dx \right)^{1/p}. \quad (2.1)$$

We also define the space $W_0^{k,p}(\Omega,\omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.1). We have that the spaces $W^{k,p}(\Omega,\omega)$ and $W_0^{k,p}(\Omega,\omega)$ are Banach spaces (see Proposition 2.1.2 in [11]).

The dual space of $W_0^{1,p}(\Omega,\omega)$ is the space $[W_0^{1,p}(\Omega,\omega)]^* = W^{-1,p'}(\Omega,\omega)$,

$$W^{-1,p'}(\Omega,\omega) = \{T = f - \operatorname{div}(G) : G = (g_1, ..., g_N), \frac{f}{\omega}, \frac{g_j}{\omega} \in L^{p'}(\Omega,\omega)\}.$$

It is evident that a weight function ω which satisfies $0 < C_1 \le \omega(x) \le C_2$, for a.e. $x \in \Omega$, gives nothing new (the space $\mathrm{W}^{k,p}(\Omega,\omega)$ is then identical with the classical Sobolev space $\mathrm{W}^{k,p}(\Omega)$). Consequently, we shall be interested in all above such weight functions ω which either vanish somewhere in $\Omega \cup \partial\Omega$ or increase to infinity (or both).

We need the following basics results.

Theorem 2.3. *(The weighted Sobolev inequality) Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let ω be an A_p-weight, $1 < p < \infty$. Then there exists positive constants C_Ω and δ such that for all $u \in W_0^{1,p}(\Omega,\omega)$ and $1 \le \eta \le N/(N-1) + \delta$*

$$\|u\|_{L^{\eta p}(\Omega,\omega)} \le C_\Omega \| |\nabla u| \|_{L^p(\Omega,\omega)}, \quad (2.2)$$

where C_Ω may be taken to depend only on N, the A_p- constant of ω, p and the diameter of Ω.

Proof. Its suffices to prove the inequality for functions $u \in C_0^\infty(\Omega)$ (see Theorem 1.3 in [4]). To extend the estimates (2.2) to arbitrary $u \in W_0^{1,p}(\Omega,\omega)$, we let $\{u_m\}$ be a sequence of $C_0^\infty(\Omega)$ functions tending to u in $W_0^{1,p}(\Omega,\omega)$. Applying the estimates (2.2) to differences $u_{m_1} - u_{m_2}$, we see that $\{u_m\}$ will be a Cauchy sequence in $L^p(\Omega,\omega)$. Consequently the limit function u will lie in the desired spaces and satisfy (2.2). □

Lemma 2.4. *(a) Let $1 < p < \infty$, then exists a constant $C_p > 0$ such that for all $\xi, \eta \in \mathbb{R}^N$,*

$$\left| |\xi|^{p-2} \xi - |\eta|^{p-2} \eta \right| \le C_p\, |\xi - \eta| (\, |\xi| + |\eta|)^{p-2}.$$

(b) Let $1 < p < \infty$. There exist two positive constants α_p and β_p such that for every $\xi, \eta \in \mathbb{R}^N$ $(N \ge 1)$

$$\alpha_p (\, |\xi| + |\eta|)^{p-2} |\xi - \eta|^2 \le \langle\, |\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta \rangle \le \beta_p (\, |\xi| + |\eta|)^{p-2} |\xi - \eta|,$$

where $\langle .,. \rangle$ denotes here the Euclidiean scalar product in $\mathbb{R}^N$.

Proof. See Proposition 17.2 and Proposition 17.3 in [2]. □

3 Weak Solutions

Let $\omega \in A_p$, $1 < p < \infty$. We denote by $X = W^{2,p}(\Omega,\omega) \cap W_0^{1,p}(\Omega,\omega)$ with the norm

$$\|u\|_X = \left(\int_\Omega |\nabla u|^p \,\omega\, dx + \int_\Omega |\Delta u|^p \,\omega\, dx \right)^{1/p}.$$

In this section we prove the existence and uniqueness of weak solutions $u \in X$ to the Navier problem

$$(P) \begin{cases} Lu(x) = f(x) - \operatorname{div}(G(x)), \quad \text{in } \Omega, \\ u(x) = \Delta u = 0, \quad \text{in } \partial\Omega, \end{cases}$$

where Ω is a bounded open set of $\mathbb{R}^N$ $(N \geq 2)$, $\dfrac{f}{\omega} \in L^{p'}(\Omega,\omega)$ and $\dfrac{G}{\nu} \in [L^{q'}(\Omega,\nu)]^N$, $G = (g_1, ..., g_N)$.

Definition 3.1. We say that $u \in X$ is a weak solution for problem (P) if

$$\begin{aligned} &\int_\Omega |\Delta u|^{p-2} \Delta u \, \Delta\varphi \,\omega\, dx + \int_\Omega |\Delta u|^{q-2} \Delta u \, \Delta\varphi \,\nu\, dx \\ &+ \int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla\varphi \rangle \,\omega\, dx + \int_\Omega |\nabla u|^{q-2} \langle \nabla u, \nabla\varphi \rangle \,\nu\, dx \\ &= \int_\Omega f\,\varphi\, dx + \int_\Omega \langle G, \nabla\varphi \rangle \, dx, \end{aligned} \tag{3.1}$$

for all $\varphi \in X$, with $f/\omega \in L^{p'}(\Omega,\omega)$ and $G/\nu \in [L^{q'}(\Omega,\nu)]^N$, where $\langle .,. \rangle$ denotes here the Euclidean scalar product in $\mathbb{R}^N$.

Remark 3.1. (i) Since $1 < q < p < \infty$ and if $\dfrac{\nu}{\omega} \in L^{p/(p-q)}(\Omega,\omega)$, there exists a constant $C_{p,q} > 0$ such that

$$\|u\|_{L^q(\Omega,\nu)} \leq C_{p,q} \|u\|_{L^p(\Omega,\omega)}, \tag{3.2}$$

where $C_{p,q} = \left[\displaystyle\int_\Omega \left(\frac{\nu}{\omega} \right)^{p/(p-q)} \omega\, dx \right]^{(p-q)/p\,q} = \|\nu/\omega\|^{1/q}_{L^{p/(p-q)}(\Omega,\omega)}$.

In fact, since $1 < q < p < \infty$, we have $r = p/q > 1$ and $r' = p/(p-q)$,

$$\begin{aligned}\|u\|^q_{L^q(\Omega,\nu)} &= \int_\Omega |u|^q\,\nu\,dx = \int_\Omega |u|^q\,\frac{\nu}{\omega}\,\omega\,dx \\ &\le \left(\int_\Omega |u|^{q\,r}\,\omega\,dx\right)^{1/r}\left(\int_\Omega \left(\frac{\nu}{\omega}\right)^{r'}\omega\,dx\right)^{1/r'} \\ &= \left(\int_\Omega |u|^p\,\omega\,dx\right)^{q/p}\left(\int_\Omega \left(\frac{\nu}{\omega}\right)^{p/(p-q)}\omega\,dx\right)^{(p-q)/p}.\end{aligned}$$

Hence, $\|u\|_{L^q(\Omega,\nu)} \le C_{p,q}\,\|u\|_{L^p(\Omega,\omega)}$.
(ii) By (3.2), we have

$$\begin{aligned}\left|\int_\Omega |\Delta u|^{q-2}\,\Delta u\,\Delta\varphi\,\nu\,dx\right| &\le \int_\Omega |\Delta u|^{q-1}\,|\Delta\varphi|\,\nu\,dx \\ &\le \left(\int_\Omega |\Delta u|^{(q-1)q'}\nu\,dx\right)^{1/q'}\left(\int_\Omega |\Delta\varphi|^q\,\nu\,dx\right)^{1/q} \\ &= \left(\int_\Omega |\Delta u|^q\,\nu\,dx\right)^{(q-1)/q}\left(\int_\Omega |\Delta\varphi|^q\,\nu\,dx\right)^{1/q} \\ &= \|\Delta u\|^{q-1}_{L^q(\Omega,\nu)}\,\|\Delta\varphi\|_{L^q(\Omega,\nu)} \\ &\le C^{q-1}_{p,q}\,\|\Delta u\|^{q-1}_{L^p(\Omega,\omega)}\,C_{p,q}\,\|\Delta\varphi\|_{L^p(\Omega,\omega)} \\ &\le C^q_{p,q}\,\|u\|^{q-1}_X\,\|\varphi\|_X,\end{aligned}$$

and, analogously, we also have

$$\begin{aligned}\left|\int_\Omega |\nabla u|^{q-2}\,\langle\nabla u,\nabla\varphi\rangle\,\nu\,dx\right| &\le \int_\Omega |\nabla u|^{q-1}\,|\nabla\varphi|\,\nu\,dx \\ &\le C^q_{p,q}\|u\|^{q-1}_X\,\|\varphi\|_X.\end{aligned}$$

Theorem 3.2. *(a) Let $\omega \in A_p$, $\nu \in \mathcal{W}(\Omega)$, $1 < q < p < \infty$ and $\dfrac{\nu}{\omega} \in L^{p/(p-q)}(\Omega,\omega)$;*
(b) $f/\omega \in L^{p'}(\Omega,\omega)$ and $G/\nu \in [L^{q'}(\Omega,\nu)]^N$.
Then the problem (P) has a unique solution $u \in X$ and

$$\|u\|_X \le \left[C_\Omega\left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} + C_{p,q}\left\|\frac{|G|}{\nu}\right\|_{L^{q'}(\Omega,\nu)}\right]^{1/(p-1)},$$

where C_Ω is the constant in Theorem 2.3 and $C_{p,q}$ is the constant in Remark 3.1 (i).

Proof. (I) *Existence.* By Theorem 2.3 (with $\eta = 1$), we have that

$$\begin{aligned}
\left|\int_\Omega f\,\varphi\,dx\right| &\leq \left(\int_\Omega \left|\frac{f}{\omega}\right|^{p'}\omega\,dx\right)^{1/p'}\left(\int_\Omega |\,\varphi|^p\,\omega\,dx\right)^{1/p} \\
&\leq C_\Omega \left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} \|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega)} \\
&\leq C_\Omega \left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} \|\varphi\|_X, \qquad (3.3)
\end{aligned}$$

and by Remark 3.1 (i)

$$\begin{aligned}
\left|\int_\Omega \langle G, \nabla\varphi\rangle\,dx\right| dx &\leq \int_\Omega |\langle G, \nabla\varphi\rangle|\,dx \\
&\leq \int_\Omega |G||\nabla\varphi|\,dx \\
&= \int_\Omega \frac{|G|}{\nu}|\nabla\varphi|\,\nu\,dx \\
&\leq \left\|\frac{|G|}{\nu}\right\|_{L^{q'}(\Omega,\nu)} \|\,|\nabla\varphi|\,\|_{L^q(\Omega,\nu)} \\
&\leq C_{p,q}\left\|\frac{|G|}{\nu}\right\|_{L^{q'}(\Omega,\nu)} \|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega)} \\
&\leq C_{p,q}\left\|\frac{|G|}{\nu}\right\|_{L^{q'}(\Omega,\nu)} \|\varphi\|_X. \qquad (3.4)
\end{aligned}$$

Define the functional $J : X \to \mathbb{R}$ by

$$\begin{aligned}
J(\varphi) &= \frac{1}{p}\int_\Omega |\Delta\varphi|^p\,\omega\,dx + \frac{1}{q}\int_\Omega |\Delta\varphi|^q\,\nu\,dx \\
&+ \frac{1}{p}\int_\Omega |\nabla\varphi|^p\,\omega\,dx + \frac{1}{q}\int_\Omega |\nabla\varphi|^q\,\nu\,dx - \int_\Omega f\,\varphi\,dx - \int_\Omega \langle G, \nabla\varphi\rangle\,dx.
\end{aligned}$$

Using (3.3), (3.4), Remark 3.1(i) and Young's inequality, we have that

$$
\begin{aligned}
J(\varphi) \geq\ & \frac{1}{p}\int_\Omega |\Delta\varphi|^p\,\omega\,dx + \frac{1}{q}\int_\Omega |\Delta\varphi|^q\,\nu\,dx \\
+\ & \frac{1}{p}\int_\Omega |\nabla\varphi|^p\,\omega\,dx + \frac{1}{q}\int_\Omega |\nabla\varphi|^q\,\nu\,dx \\
-\ & \left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} \|\varphi\|_{L^p(\Omega,\omega)} - \left\|\frac{|G|}{\nu}\right\|_{L^{q'}(\Omega,\nu)} \|\,|\nabla\varphi|\,\|_{L^q(\Omega,\nu)} \\
\geq\ & \frac{1}{p}\int_\Omega |\nabla\varphi|^p\,\omega\,dx + \frac{1}{q}\int_\Omega |\nabla\varphi|^q\,\nu\,dx \\
-\ & C_\Omega\left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} \|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega)} - \left\|\frac{|G|}{\nu}\right\|_{L^{q'}(\Omega,\nu)} \|\,|\nabla\varphi|\,\|_{L^q(\Omega,\nu)} \\
\geq\ & \frac{1}{p}\int_\Omega |\nabla\varphi|^p\,\omega\,dx + \frac{1}{q}\int_\Omega |\nabla\varphi|^q\,\nu\,dx \\
-\ & \frac{C_\Omega^{p'}}{p'}\left\|\frac{f}{\omega}\right\|^{p'}_{L^{p'}(\Omega,\omega)} - \frac{1}{p}\|\,|\nabla\varphi|\,\|^p_{L^p(\Omega,\omega)} - \frac{1}{q'}\left\|\frac{|G|}{\nu}\right\|^{q'}_{L^{q'}(\Omega,\nu)} - \frac{1}{q}\|\,|\nabla\varphi|\,\|^q_{L^q(\Omega,\nu)} \\
\geq\ & -\frac{C_\Omega^{p'}}{p'}\left\|\frac{f}{\omega}\right\|^{p'}_{L^{p'}(\Omega,\omega)} - \frac{1}{q'}\left\|\frac{|G|}{\nu}\right\|^{q'}_{L^{q'}(\Omega,\nu)}
\end{aligned}
$$

that is, J is bounded from below. Let $\{u_n\}$ be a minimizing sequence, that is, a sequence such that

$$J(u_n) \;\to\; \inf_{\varphi\in X} J(\varphi)\,.$$

Then for n large enough, we obtain

$$
\begin{aligned}
0 \geq J(u_n) =\ & \frac{1}{p}\int_\Omega |\Delta u_n|^p\,\omega\,dx + \frac{1}{q}\int_\Omega |\Delta u_n|^q\,\nu\,dx \\
+\ & \frac{1}{p}\int_\Omega |\nabla u_n|^p\,\omega\,dx + \frac{1}{q}\int_\Omega |\nabla u_n|^q\,\nu\,dx \\
-\ & \int_\Omega f\,u_n\,dx - \int_\Omega \langle G, \nabla u_n\rangle\,dx,
\end{aligned}
$$

and we have

$$
\begin{aligned}
& \frac{1}{p}\int_\Omega |\Delta u_n|^p\,\omega\,dx + \frac{1}{p}\int_\Omega |\nabla u_n|^p\,\omega\,dx \\
\leq & \frac{1}{p}\int_\Omega |\Delta u_n|^p\,\omega\,dx + \frac{1}{q}\int_\Omega |\Delta u_n|^q\,\nu\,dx + \frac{1}{p}\int_\Omega |\nabla u_n|^p\,\omega\,dx + \frac{1}{q}\int_\Omega |\nabla u_n|^q\,\nu\,dx \\
\leq & \int_\Omega f\,u_n\,dx + \int_\Omega \langle G, u_n\rangle\,dx. \qquad (3.5)
\end{aligned}
$$

Hence, by Theorem 2.3 (with $\eta = 1$), (3.5) and Remark 3.1(i), we obtain

$$\begin{aligned}
\|u_n\|_X^p &= \int_\Omega |\Delta u_n|^p \, \omega \, dx + \int_\Omega |\nabla u_n|^p \, \omega \, dx \\
&\le p\Big(\int_\Omega f \, u_n \, dx + \int_\Omega \langle G, \nabla u_n \rangle \, dx \Big) \\
&\le p \left(\left\| \frac{f}{\omega} \right\|_{L^{p'}(\Omega,\omega)} \|u_n\|_{L^p(\Omega,\omega)} + \left\| \frac{|G|}{\nu} \right\|_{L^{q'}(\Omega,\nu)} \| \, |\nabla u_n| \, \|_{L^q(\Omega,\nu)} \right) \\
&\le p \left(C_\Omega \left\| \frac{f}{\omega} \right\|_{L^{p'}(\Omega,\omega)} \| \, |\nabla u_n| \, \|_{L^p(\Omega,\omega)} + C_{p,q} \left\| \frac{|G|}{\nu} \right\|_{L^{q'}(\Omega,\nu)} \| \, |\nabla u_n| \, \|_{L^p(\Omega,\omega)} \right) \\
&\le p \left(C_\Omega \left\| \frac{f}{\omega} \right\|_{L^{p'}(\Omega,\omega)} + C_{p,q} \left\| \frac{|G|}{\nu} \right\|_{L^{q'}(\Omega,\nu)} \right) \|u_n\|_X .
\end{aligned}$$

Hence,

$$\|u_n\|_X \le \left[p \left(C_\Omega \left\| \frac{f}{\omega} \right\|_{L^{p'}(\Omega,\omega)} + C_{p,q} \left\| \frac{|G|}{\nu} \right\|_{L^{q'}(\Omega,\nu)} \right) \right]^{1/(p-1)} .$$

Therefore $\{u_n\}$ is bounded in X. Since X is reflexive, there exists a subsequence, still denoted by $\{u_n\}$, and a function $u \in X$ such that $u_n \rightharpoonup u$ in X. Since,

$$X \ni \varphi \mapsto \int_\Omega f \, \varphi \, dx + \int_\Omega \langle G, \nabla \varphi \rangle \, dx,$$

and

$$X \ni \varphi \mapsto \|\Delta \varphi\|_{L^p(\Omega,\omega)} + \|\Delta \varphi\|_{L^q(\Omega,\nu)} + \| \, |\nabla \varphi| \, \|_{L^p(\Omega,\omega)} + \| \, |\nabla \varphi| \, \|_{L^q(\Omega,\nu)},$$

are continuous then J is continuous. Moreover since $1 < q < p < \infty$ we have that J is convex and thus lower semi-continuous for the weak convergence. It follows that

$$J(u) \le \liminf_n J(u_n) = \inf_{\varphi \in X} J(\varphi),$$

and thus u is a minimizer of J on X (see Theorem 25.C and Corollary 25.15 in [12]). For any $\varphi \in X$ the function

$$\begin{aligned}
\lambda \mapsto \quad & \frac{1}{p} \int_\Omega |\Delta(u + \lambda\varphi)|^p \, \omega \, dx + \frac{1}{q} \int_\Omega |\Delta(u + \lambda\varphi)|^q \, \nu \, dx \\
& + \frac{1}{p} \int_\Omega |\nabla(u + \lambda \, \varphi)|^p \, \omega \, dx + \frac{1}{q} \int_\Omega |\nabla(u + \lambda\varphi)|^q \, \nu \, dx \\
& - \int_\Omega (u + \lambda \, \varphi) \, f \, dx - \int_\Omega \langle G, \nabla(u + \lambda \, \varphi) \rangle \, dx
\end{aligned}$$

has a minimum at $\lambda = 0$. Hence,

$$\frac{d}{d\lambda}\Big(J(u+\lambda\,\varphi)\Big)\Big|_{\lambda=0} = 0,\ \forall\,\varphi\in X.$$

We have

$$\frac{d}{d\,\lambda}\left(|\,\nabla(u+\lambda\,\varphi)|^p\,\omega\right) = p\,\{|\nabla(u+\lambda\,\varphi)|^{p-2}(\langle\nabla u,\nabla\varphi\rangle+\lambda\,|\nabla\varphi|^2)\}\,\omega,$$

and

$$\frac{d}{d\,\lambda}\left(|\,\Delta(u+\lambda\,\varphi)|^p\,\omega\right) = p\,|\Delta u+\lambda\Delta\varphi|^{p-2}(\,\Delta u+\lambda\Delta\varphi)\,\Delta\varphi\,\omega,$$

and we obtain

$$\begin{aligned}
0 &= \frac{d}{d\lambda}\Big(J(u+\lambda\,\varphi)\Big)\Big|_{\lambda=0}\\
&= \Bigg[\frac{1}{p}\Bigg(p\int_\Omega|\nabla(u+\lambda\,\varphi)|^{p-2}(\langle\nabla u,\nabla\varphi\rangle+\lambda\,|\nabla\varphi|^2)\,\omega\,dx\\
&+ p\int_\Omega|\Delta u+\lambda\Delta\varphi|^{p-2}(\,\Delta u+\lambda\Delta\varphi)\,\Delta\varphi\,\omega\,dx\Bigg)\\
&+ \frac{1}{q}\Bigg(q\int_\Omega|\nabla(u+\lambda\,\varphi)|^{q-2}(\langle\nabla u,\nabla\varphi\rangle+\lambda\,|\nabla\varphi|^2)\,\nu\,dx\\
&+ q\int_\Omega|\Delta u+\lambda\Delta\varphi|^{q-2}(\,\Delta u+\lambda\Delta\varphi)\,\Delta\varphi\,\nu\,dx\Bigg)\\
&- \int_\Omega\varphi\,f\,dx-\int_\Omega\langle G,\nabla\varphi\rangle\,dx\Bigg]\Bigg|_{\lambda=0}\\
&= \int_\Omega|\Delta u|^{p-2}\Delta u\,\Delta\varphi\,\omega\,dx+\int_\Omega|\nabla u|^{p-2}\,\langle\nabla u,\nabla\varphi\rangle\,\omega\,dx\\
&+ \int_\Omega|\Delta u|^{q-2}\Delta u\,\Delta\varphi\,\nu\,dx+\int_\Omega|\nabla u|^{q-2}\,\langle\nabla u,\nabla\varphi\rangle\,\nu\,dx\\
&- \int_\Omega f\,\varphi\,dx-\int_\Omega\langle G,\nabla\varphi\rangle\,dx.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_\Omega|\Delta u|^{p-2}\Delta u\,\Delta\varphi\,\omega\,dx+\int_\Omega|\nabla u|^{p-2}\langle\nabla u\,\nabla\varphi\rangle\,\omega\,dx\\
&+\int_\Omega|\Delta u|^{q-2}\Delta u\,\Delta\varphi\,\nu\,dx+\int_\Omega|\nabla u|^{q-2}\,\langle\nabla u,\nabla\varphi\rangle\,\nu\,dx\\
&=\int_\Omega f\,\varphi\,dx+\int_\Omega\langle G,\nabla\varphi\rangle\,dx,
\end{aligned}$$

for all $\varphi \in X$, that is, $u \in X$ is a solution of problem (P).

(II) *Uniqueness.* If $u_1, u_2 \in X$ are two weak solutions of problem (P), we have

$$\begin{aligned}
&\int_\Omega |\Delta u_1|^{p-2} \Delta u_1 \, \Delta\varphi \, \omega \, dx + \int_\Omega |\Delta u_1|^{q-2} \Delta u_1 \, \Delta\varphi \, \nu \, dx \\
&+ \int_\Omega |\nabla u_1|^{p-2} \langle \nabla u_1, \nabla\varphi \rangle \, \omega \, dx + \int_\Omega |\nabla u_1|^{q-2} \langle \nabla u_1, \nabla\varphi \rangle \, \nu \, dx \\
&= \int_\Omega f \, \varphi \, dx + \int_\Omega \langle G, \nabla\varphi \rangle \, dx,
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega |\Delta u_2|^{p-2} \Delta u_2 \, \Delta\varphi \, \omega \, dx + \int_\Omega |\Delta u_2|^{q-2} \Delta u_2 \, \Delta\varphi \, \nu \, dx \\
&+ \int_\Omega |\nabla u_2|^{p-2} \langle \nabla u_2, \nabla\varphi \rangle \, \omega \, dx + \int_\Omega |\nabla u_2|^{q-2} \langle \nabla u_2, \nabla\varphi \rangle \, \nu \, dx \\
&= \int_\Omega f \, \varphi \, dx + \int_\Omega \langle G, \nabla\varphi \rangle \, dx,
\end{aligned}$$

for all $\varphi \, X$. Hence

$$\begin{aligned}
&\int_\Omega \Big(|\Delta u_1|^{p-2} \Delta u_1 - |\Delta u_2|^{p-2} \Delta u_2 \Big) \Delta\varphi \, \omega \, dx \\
&+ \int_\Omega \Big(|\Delta u_1|^{q-2} \Delta u_1 - |\Delta u_2|^{q-2} \Delta u_2 \Big) \Delta\varphi \, \nu \, dx \\
&+ \int_\Omega \Big(|\nabla u_1|^{p-2} \langle \nabla u_1, \nabla\varphi \rangle - |\nabla u_2|^{p-2} \langle \nabla u_2, \nabla\varphi \rangle \Big) \, \omega \, dx \\
&+ \int_\Omega \Big(|\nabla u_1|^{q-2} \langle \nabla u_1, \nabla\varphi \rangle - |\nabla u_2|^{q-2} \langle \nabla u_2, \nabla\varphi \rangle \Big) \, \nu \, dx = 0.
\end{aligned}$$

Taking $\varphi = u_1 - u_2$, and using Lemma 2.4(b) there exist positive constants $\alpha_p, \tilde{\alpha}_p, \alpha_q, \tilde{\alpha}_q$ such that

$$\begin{aligned}
0 &= \int_\Omega \Big(|\Delta u_1|^{p-2}\Delta u_1 - |\Delta u_2|^{p-2}\Delta u_2\Big)(\,\Delta u_1 - \Delta u_2)\,\omega\,dx \\
&+ \int_\Omega \Big(|\Delta u_1|^{q-2}\Delta u_1 - |\Delta u_2|^{q-2}\Delta u_2\Big)(\,\Delta u_1 - \Delta u_2)\,\nu\,dx \\
&+ \int_\Omega \Big(|\nabla u_1|^{p-2}\langle\nabla u_1, \nabla u_1 - \nabla u_2\rangle - |\nabla u_2|^{p-2}\langle\nabla u_2, \nabla u_1 - \nabla u_2\rangle\Big)\,\omega\,dx \\
&+ \int_\Omega \Big(|\nabla u_1|^{q-2}\langle\nabla u_1, \nabla u_1 - \nabla u_2\rangle - |\nabla u_2|^{q-2}\langle\nabla u_2, \nabla u_1 - \nabla u_2\rangle\Big)\,\nu\,dx \\
&= \int_\Omega \Big(|\Delta u_1|^{p-2}\Delta u_1 - |\Delta u_2|^{p-2}\Delta u_2\Big)(\,\Delta u_1 - \Delta u_2)\,\omega\,dx
\end{aligned}$$

$$\begin{aligned}
&+ \int_\Omega \Big(|\Delta u_1|^{q-2}\Delta u_1 - |\Delta u_2|^{q-2}\Delta u_2\Big)(\,\Delta u_1 - \Delta u_2)\,\nu\,dx \\
&+ \int_\Omega \langle\,|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2, \nabla u_1 - \nabla u_2\rangle\,\omega\,dx \\
&+ \int_\Omega \langle\,|\nabla u_1|^{q-2}\nabla u_1 - |\nabla u_2|^{q-2}\nabla u_2, \nabla u_1 - \nabla u_2\rangle\,\nu\,dx \\
&\geq \alpha_p \int_\Omega \Big(|\Delta u_1| + |\Delta u_2|\Big)^{p-2} |\Delta u_1 - \Delta u_2|^2\,\omega\,dx \\
&+ \tilde{\alpha}_p \int_\Omega \Big(|\nabla u_1| + |\nabla u_2|\Big)^{p-2} |\nabla u_1 - \nabla u_2|^2\,\omega\,dx \\
&+ \alpha_q \int_\Omega \Big(|\Delta u_1| + |\Delta u_2|\Big)^{q-2} |\Delta u_1 - \Delta u_2|^2\,\nu\,dx \\
&+ \tilde{\alpha}_q \int_\Omega \Big(|\nabla u_1| + |\nabla u_2|\Big)^{q-2} |\nabla u_1 - \nabla u_2|^2\,\nu\,dx \\
&\geq \alpha_p \int_\Omega \Big(|\Delta u_1| + |\Delta u_2|\Big)^{p-2} |\Delta u_1 - \Delta u_2|^2\,\omega\,dx \\
&+ \tilde{\alpha}_p \int_\Omega \Big(|\nabla u_1| + |\nabla u_2|\Big)^{p-2} |\nabla u_1 - \nabla u_2|^2\,\omega\,dx
\end{aligned}$$

Therefore $\Delta u_1 = \Delta u_2$ and $\nabla u_1 = \nabla u_2$ a.e. and since $u_1, u_2 \in X$, then $u_1 = u_2$ a.e. (by Remark 2.1).

(III) *Estimate for* $\|u\|_X$.

In particular, for $\varphi = u \in X$ in Definition 3.1 we have

$$\int_\Omega |\Delta u|^p\,\omega\,dx + \int_\Omega |\Delta u|^q\,\nu\,dx + \int_\Omega |\nabla u|^p\,\omega\,dx + \int_\Omega |\nabla u|^q\,\nu\,dx$$
$$= \int_\Omega f\,u\,dx + \int_\Omega \langle G, \nabla u\rangle\,dx.$$

Then, by Theorem 2.3 and Remark 3.1 (i), we obtain

$$\begin{aligned}\|u\|_X^p &= \int_\Omega |\Delta u|^p\,\omega\,dx + \int_\Omega |\nabla u|^p\,\omega\,dx\\ &\leq \int_\Omega |\Delta u|^p\,\omega\,dx + \int_\Omega |\Delta u|^q\,\nu\,dx + \int_\Omega |\nabla u|^p\,\omega\,dx + \int_\Omega |\nabla u|^q\,\nu\,dx\end{aligned}$$

$$\begin{aligned}&= \int_\Omega f\,u\,dx + \int_\Omega \langle G, \nabla u\rangle\,dx\\ &\leq \left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} \|u\|_{L^p(\Omega,\omega)} + \left\|\frac{|G|}{\nu}\right\|_{L^{q'}(\Omega,\nu)} \|\,|\nabla u|\,\|_{L^q(\Omega,\nu)}\\ &\leq C_\Omega\left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} \|\,|\nabla u|\,\|_{L^p(\Omega,\omega)} + C_{p,q}\left\|\frac{|G|}{\nu}\right\|_{L^{q'}(\Omega,\nu)} \|\,|\nabla u|\,\|_{L^p(\Omega,\omega)}\\ &\leq \left(C_\Omega\left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} + C_{p,q}\left\|\frac{|G|}{\nu}\right\|_{L^{q'}(\Omega,\nu)}\right)\|u\|_X.\end{aligned}$$

Therefore,

$$\|u\|_X \leq \left(C_\Omega\left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} + C_{p,q}\left\|\frac{|G|}{\nu}\right\|_{L^{q'}(\Omega,\nu)}\right)^{1/(p-1)}.$$

□

Corollary 3.3. *Under the assumptions of Theorem 3.2 with* $2 \leq q < p < \infty$*. If* $u_1, u_2 \in X$ *are solutions of*

$$(P_1)\begin{cases} Lu_1(x) = f(x) - \mathrm{div}(G(x)), & \text{in } \Omega,\\ u_1(x) = \Delta u_1(x) = 0, & \text{in } \partial\Omega,\end{cases}$$

and

$$(P_2)\begin{cases} Lu_2(x) = \tilde{f}(x) - \mathrm{div}(\tilde{G}(x)), & \text{in } \Omega,\\ u_2(x) = \Delta u_2(x) = 0, & \text{in } \partial\Omega,\end{cases}$$

then

$$\|u_1 - u_2\|_X \leq \frac{1}{\gamma^{1/(p-1)}}\left(C_\Omega\left\|\frac{f-\tilde{f}}{\omega}\right\|_{L^{p'}(\Omega,\omega)} + C_{p,q}\left\|\frac{|G-\tilde{G}|}{\nu}\right\|_{L^{q'}(\Omega,\nu)}\right)^{1/(p-1)},$$

where γ *is a positive constant,* C_Ω *and* $C_{p,q}$ *are the same constants of Theorem 3.2.*

Proof. If u_1 and u_2 are solutions of $(P1)$ and $(P2)$ then for all $\varphi\in X$ we have

$$\begin{aligned}
&\int_\Omega |\Delta u_1|^{p-2}\Delta u_1\,\Delta\varphi\,\omega\,dx + \int_\Omega |\Delta u_1|^{q-2}\Delta u_1\,\Delta\varphi\,\nu\,dx\\
&+\int_\Omega |\nabla u_1|^{p-2}\langle\nabla u_1,\nabla\varphi\rangle\,\omega\,dx + \int_\Omega |\nabla u_1|^{q-2}\,\langle\nabla u_1,\nabla\varphi\rangle\,\nu\,dx\\
&-\Big(\int_\Omega |\Delta u_2|^{p-2}\Delta u_2\,\Delta\varphi\,\omega\,dx + \int_\Omega |\Delta u_2|^{q-2}\Delta u_2\,\Delta\varphi\,\nu\,dx\\
&+\int_\Omega |\nabla u_1|^{p-2}\langle\nabla u_2,\nabla\varphi\rangle\,\omega\,dx + \int_\Omega |\nabla u_2|^{q-2}\,\langle\nabla u_2,\nabla\varphi\rangle\,\nu\,dx\Big)\\
= &\int_\Omega (f-\tilde f)\,\varphi\,dx + \int_\Omega \langle G-\tilde G,\nabla\varphi\rangle\,dx.
\end{aligned} \tag{3.6}$$

In particular, for $\varphi = u_1 - u_2$, we obtain

(i) Since $2\leq q < p < \infty$ and by Lemma 2.4 (b), there exist two positive constants α_p and α_q such that

$$\begin{aligned}
&\int_\Omega \Big(|\Delta u_1|^{p-2}\Delta u_1 - |\Delta_2|^{p-2}\Delta u_2\Big)\,\Delta(u_1-u_2)\,\omega\,dx\\
&\geq \alpha_p \int_\Omega \Big(|\Delta u_1| + |\Delta u_2|\Big)^{p-2}\,|\Delta u_1 - \Delta u_2|^2\,\omega\,dx\\
&\geq \alpha_p \int_\Omega |\Delta u_1 - \Delta u_2|^{p-2}|\Delta u_1 - \Delta u_2|^2\,\omega\,dx = \alpha_p \int_\Omega |\Delta(u_1-u_2)|^p\,\omega\,dx,
\end{aligned}$$

and analogously

$$\int_\Omega \Big(|\Delta u_1|^{q-2}\Delta u_1 - |\Delta u_2|^{q-2}\Delta u_2\Big)\,\Delta(u_1-u_2)\,\nu\,dx \geq \alpha_q \int_\Omega |\Delta(u_1-u_2)|^q\,\nu\,dx \geq 0.$$

(ii) Since $2\leq q < p < \infty$ and by Lemma 2.4 (b), there exit two positive constants $\tilde\alpha_p$ and $\tilde\alpha_q$ such that

$$\begin{aligned}
&\int_\Omega \Big(|\nabla u_1|^{p-2}\langle\nabla u_1,\nabla(u_1-u_2)\rangle - |\nabla u_2|^{p-2}\langle\nabla u_2,\nabla(u_1-u_2)\rangle\Big)\,\omega\,dx\\
&= \int_\Omega \langle|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2, \nabla(u_1-u_2)\rangle\,\omega\,dx\\
&\geq \tilde\alpha_p \int_\Omega (|\nabla u_1| + |\nabla u_2|)^{p-2}|\nabla u_1 - \nabla u_2|^2\,\omega\,dx\\
&\geq \tilde\alpha_p \int_\Omega |\nabla u_1 - \nabla u_2|^{p-2}\,|\nabla u_1 - \nabla u_2|^2\,\omega\,dx = \tilde\alpha_p \int_\Omega |\nabla(u_1-u_2)|^p\,\omega\,dx,
\end{aligned}$$

and analogously,

$$\int_\Omega \left(|\nabla u_1|^{q-2} \langle \nabla u_1, \nabla(u_1 - u_2)\rangle - |\nabla u_2|^{q-2} \langle \nabla u_2, \nabla(u_1 - u_2)\rangle \right) \nu \, dx$$
$$\geq \tilde{\alpha}_q \int_\Omega |\nabla(u_1 - u_2)|^q \, \nu \, dx \geq 0.$$

(iii) By Remark 3.1 (i) we have

$$\left| \int_\Omega (f - \tilde{f})\,(u_1 - u_2)\, dx + \int_\Omega \langle G - \tilde{G}, \nabla(u_1 - u_2)\rangle \, dx \right|$$
$$\leq \left(C_\Omega \left\| \frac{f - \tilde{f}}{\omega} \right\|_{L^{p'}(\Omega,\omega)} + C_{p,q} \left\| \frac{|G - \tilde{G}|}{\nu} \right\|_{L^{q'}(\Omega,\nu)} \right) \|u_1 - u_2\|_X .$$

Hence, with $\gamma = \min\{\alpha_p, \tilde{\alpha}_p\}$, we obtain in (3.6)

$$\gamma \, \|u_1 - u_2\|_X^p \leq \alpha_p \int_\Omega |\Delta(u_1 - u_2)|^p \, \omega \, dx + \tilde{\alpha}_p \int_\Omega |\nabla(u_1 - u_2)|^p \, \omega \, dx$$
$$\leq \left(C_\Omega \left\| \frac{f - \tilde{f}}{\omega} \right\|_{L^{p'}(\Omega,\omega)} + C_{p,q} \left\| \frac{|G - \tilde{G}|}{\nu} \right\|_{L^{q'}(\Omega,\nu)} \right) \|u_1 - u_2\|_X .$$

Therefore,

$$\|u_1 - u_2\|_X \leq \frac{1}{\gamma^{1/(p-1)}} \left(C_\Omega \left\| \frac{f - \tilde{f}}{\omega} \right\|_{L^{p'}(\Omega,\omega)} + C_{p,q} \left\| \frac{|G - \tilde{G}|}{\nu} \right\|_{L^{q'}(\Omega,\nu)} \right)^{1/(p-1)} .$$

□

Corollary 3.4. *Assume $2 \leq q < p < \infty$. Let the assumptions of Theorem 3.2 be fulfilled, and let $\{f_m\}$ and $\{G_m\}$ be sequences of functions satisfying $\frac{f_m}{\omega} \to \frac{f}{\omega}$ in $L^{p'}(\Omega, \omega)$ and $\left\| \frac{|G_m - G|}{\nu} \right\|_{L^{q'}(\Omega,\nu)} \to 0$ as $m \to \infty$. If $u_m \in X$ is a solution of the problem*

$$(P_m) \begin{cases} Lu_m(x) = f_m(x) - \operatorname{div}(G_m(x)), & \text{in } \Omega, \\ u_m(x) = \Delta u_m(x) = 0, & \text{in } \partial\Omega, \end{cases}$$

then $u_m \to u$ in X and u is a solution of problem (P).

Proof. By Corollary 3.3 we have

$$\|u_m - u_r\|_X \leq \frac{1}{\gamma^{1/(p-1)}} \left(C_\Omega \left\| \frac{f_m - f_r}{\omega} \right\|_{L^{p'}(\Omega,\omega)} + C_{p,q} \left\| \frac{|G_m - G_r|}{\nu} \right\|_{L^{q'}(\Omega,\nu)} \right)^{1/(p-1)} .$$

Therefore $\{u_m\}$ is a Cauchy sequence in X. Hence, there is $u\in X$ such that $u_m \to u$ in X. We have that u is a solution of problem (P). In fact, since u_m is a solution of (P_m), for all $\varphi\in X$ we have

$$\begin{aligned}
&\int_\Omega |\Delta u|^{p-2}\Delta u\,\Delta\varphi\,\omega\,dx + \int_\Omega |\Delta u|^{q-2}\Delta u\,\Delta\varphi\,\nu\,dx\\
&+\int_\Omega |\nabla u|^{p-2}\langle\nabla u,\nabla\varphi\rangle\,\omega\,dx + \int_\Omega |\nabla u|^{q-2}\langle\nabla u,\nabla\varphi\rangle\,\nu\,dx\\
&=\int_\Omega \Big(|\Delta u|^{p-2}\Delta u - |\Delta_m|^{p-2}\Delta u_m\Big)\,\Delta\varphi\,\omega\,dx\\
&+\int_\Omega \Big(|\Delta u|^{q-2}\Delta u - |\Delta u_m|^{q-2}\Delta u_m\Big)\Delta\varphi\,\nu\,dx\\
&+\int_\Omega \Big(|\nabla u|^{p-2}\langle\nabla u,\nabla\varphi\rangle - |\nabla u_m|^{p-2}\langle\nabla u_m,\nabla\varphi\rangle\Big)\,\omega\,dx\\
&+\int_\Omega \Big(|\nabla u|^{q-2}\langle\nabla u,\nabla\varphi\rangle - |\nabla u_m|^{q-2}\langle\nabla u_m,\nabla\varphi\rangle\Big)\,\nu\,dx\\
&+\int_\Omega |\Delta u_m|^{p-2}\Delta u_m\,\Delta\varphi\,\omega\,dx + \int_\Omega |\Delta u_m|^{q-2}\Delta u_m\,\Delta\varphi\,\nu\,dx\\
&+\int_\Omega |\nabla u_m|^{p-2}\langle\nabla u_m,\nabla\varphi\rangle\,\omega\,dx + \int_\Omega |\nabla u_m|^{q-2}\langle\nabla u_m,\nabla\varphi\rangle\,\nu\,dx\\
&= I_1+I_2+I_3+I_4+\int_\Omega f_m\varphi\,dx + \int_\Omega \langle G_m,\nabla\varphi\rangle\,dx\\
&= I_1+I_2+I_3+I_4+\int_\Omega f\,\varphi\,dx + \int_\Omega \langle G,\nabla\varphi\rangle\,dx\\
&+\int_\Omega (f_m-f)\,\varphi\,dx + \int_\Omega \langle G_m-G,\nabla\varphi\rangle\,dx, \qquad (3.7)
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \int_\Omega \Big(|\Delta u|^{p-2}\Delta u - |\Delta u_m|^{p-2}\Delta u_m\Big)\,\Delta\varphi\,\omega\,dx,\\
I_2 &= \int_\Omega \Big(|\Delta u|^{q-2}\Delta u - |\Delta u_m|^{q-2}\Delta u_m\Big)\Delta\varphi\,\nu\,dx,\\
I_3 &= \int_\Omega \Big(|\nabla u|^{p-2}\langle\nabla u,\nabla\varphi\rangle - |\nabla u_m|^{p-2}\langle\nabla u_m,\nabla\varphi\rangle\Big)\,\omega\,dx,\\
I_4 &= \int_\Omega \Big(|\nabla u|^{q-2}\langle\nabla u,\nabla\varphi\rangle - |\nabla u_m|^{q-2}\langle\nabla u_m,\nabla\varphi\rangle\Big)\,\nu\,dx.
\end{aligned}$$

We have that:

(1) By Lemma 2.4 (a) there exists $C_p > 0$ such that

$$\begin{aligned} |I_1| &\leq \int_\Omega \big||\Delta u|^{p-2}\Delta u - |\Delta u_m|^{p-2}\Delta u_m\big|\,|\Delta\varphi|\,\omega\,dx \\ &\leq C_p \int_\Omega |\Delta u - \Delta u_m|\,(|\Delta u| + |\Delta u_m|)^{p-2}\,|\Delta\varphi|\,\omega\,dx. \end{aligned}$$

Let $r = p/(p-2)$. Since $\dfrac{1}{p} + \dfrac{1}{p} + \dfrac{1}{r} = 1$, by the Generalized Hölder inequality we obtain

$$\begin{aligned} &|I_1| \\ &\leq C_p\left(\int_\Omega |\Delta u - \Delta u_m|^p\,\omega\,dx\right)^{1/p}\left(\int_\Omega |\Delta\varphi|^p\,\omega\,dx\right)^{1/p}\left(\int_\Omega (|\Delta u| + |\Delta u_m|)^{(p-2)r}\,\omega\,dx\right)^{1/r} \\ &\leq C_p\|u - u_m\|_X\,\|\varphi\|_X\||\Delta u| + |\Delta u_m|\|^{(p-2)}_{L^p(\Omega,\omega)}. \end{aligned}$$

Now, since $u_m \to u$ in X, then exists a constant $M > 0$ such that $\|u_m\|_X \leq M$. Hence,

$$\||\Delta u| + |\Delta u_m|\|_{L^p(\Omega,\omega)} \leq \|u\|_X + \|u_m\|_X \leq 2M. \tag{3.8}$$

Therefore,

$$\begin{aligned} |I_1| &\leq C_p\,(2M)^{p-2}\,\|u - u_m\|_X\,\|\varphi\|_X \\ &= C_1\,\|u - u_m\|_X\,\|\varphi\|_X. \end{aligned}$$

Analogously, there exists a constant C_3 such that

$$|I_3| \leq C_3\|u - u_m\|_X\,\|\varphi\|_X.$$

(2) By Lemma 2.4 (a) there exists a positive constant C_q such that

$$\begin{aligned} |I_2| &\leq \int_\Omega \big||\Delta u|^{q-2}\Delta u - |\Delta u_m|^{q-2}\Delta u_m\big|\,|\Delta\varphi|\,\nu\,dx \\ &\leq C_q \int_\Omega |\Delta u - \Delta u_m|\,(|\Delta u| + |\Delta u_m|)^{q-2}\,|\Delta\varphi|\,\nu\,dx. \end{aligned}$$

Let $s = q/(q-2)$ (if $2 < q < p < \infty$). Since $\dfrac{1}{q} + \dfrac{1}{q} + \dfrac{1}{s} = 1$, by the Generalized Hölder inequality we obtain

$$\begin{aligned} &|I_2| \\ &\leq C_q\left(\int_\Omega |\Delta u - \Delta u_m|^q\,\nu\,dx\right)^{1/q}\left(\int_\Omega |\Delta\varphi|^q\,\nu\,dx\right)^{1/q}\left(\int_\Omega (|\Delta u| + |\Delta u_m|)^{(q-2)s}\,\nu\,dx\right)^{1/s} \\ &= C_q\,\|\Delta u - \Delta u_m\|_{L^q(\Omega,\nu)}\,\|\Delta\varphi\|_{L^q(\Omega,\nu)}\||\Delta u| + |\Delta u_m|\|^{q-2}_{L^q(\Omega,\nu)}. \end{aligned}$$

Now, by Remark 3.1 (i) and (3.8) we have

$$\begin{aligned}|I_2| &\leq C_q\ C_{p,q}\|\Delta u-\Delta u_m\|_{L^p(\Omega,\omega)}\,C_{p,q}\|\Delta\varphi\|_{L^p(\Omega,\omega)}C_{p,q}^{q-2}\||\Delta u|+|\Delta u_m|\|_{L^p(\Omega,\omega)}^{q-2}\\ &\leq C_q\ C_{p,q}^{q}\|u-u_m\|_X\|\varphi\|_X\ (2M)^{q-2}\\ &= C_2\,\|u-u_m\|_X\ \|\varphi\|_X.\end{aligned}$$

Analogously, there exists a positive constant C_4 such that

$$|I_4|\leq C_4\,\|u-u_m\|_X\ \|\varphi\|_X.$$

In case $q=2$, we have $|I_2|,|I_4|\leq C_{p,2}^2\,\|u-u_m\|_X\,\|\varphi\|_X$.
Therefore, we have $I_1,I_2,I_3,I_4\to 0$ when $m\to\infty$.

(3) We also have

$$\begin{aligned}&\left|\int_\Omega (f_m-f)\,\varphi\,dx+\int_\Omega\langle G_m-G,\nabla\varphi\rangle\,dx\right|\\ &\left(C_\Omega\left\|\frac{f_m-f}{\omega}\right\|_{L^{p'}(\Omega,\omega)}+C_{p,q}\left\|\frac{|G_m-G|}{\nu}\right\|_{L^{q'}(\Omega,\nu)}\right)\|\varphi\|_X\\ &\to 0,\end{aligned}$$

when $m\to\infty$.

Therefore, in (3.7), we obtain when $m\to\infty$ that

$$\begin{aligned}&\int_\Omega|\Delta u|^{p-2}\Delta u\,\Delta\varphi\,\omega\,dx+\int_\Omega|\Delta u|^{q-2}\Delta u\,\Delta\varphi\,\nu\,dx\\ &+\int_\Omega|\nabla u|^{p-2}\langle\nabla u,\nabla\varphi\rangle\,\omega\,dx+\int_\Omega|\nabla u|^{q-2}\langle\nabla u,\nabla\varphi\rangle\,\nu\,dx\\ &=\int_\Omega f\,\varphi\,dx+\int_\Omega\langle G,\nabla\varphi\rangle\,dx,\end{aligned}$$

i.e., u is a solution of problem (P). □

Example Let $\Omega=\{(x,y)\in\mathbb{R}^2\,:\,x^2+y^2<1\}$, $w(x,y)=(x^2+y^2)^{-1/2}$ ($\omega\in A_4$, $p=4$ and $q=3$), $\nu(x,y)=(x^2+y^2)^{-1/3}$, $f(x,y)=\dfrac{\cos(xy)}{(x^2+y^2)^{1/6}}$ and $G(x,y)=\left(\dfrac{\sin(x+y)}{(x^2+y^2)^{1/6}},\dfrac{\sin(xy)}{(x^2+y^2)^{1/6}}\right)$. By Theorem 3.2 , the problem

$$\begin{cases}\Delta\Big[(x^2+y^2)^{-1/2}\,|\Delta u|^2\Delta u+(x^2+y^2)^{-1/3}|\Delta u|\Delta u\Big]\\ -\operatorname{div}\Big[(x^2+y^2)^{-1/2}|\nabla u|^2\nabla u+(x^2+y^2)^{-1/3}|\nabla u|\nabla u\Big]\\ =f(x)-\operatorname{div}(G(x)),\quad\text{in}\ \ \Omega\\ u(x)=\Delta u=0,\quad\text{in}\ \ \partial\Omega\end{cases}$$

has a unique solution $u\in W^{2,4}(\Omega,\omega)\cap W_0^{1,4}(\Omega,\omega)$.

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