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$*$-ARMENDARIZ PROPERTY FOR INVOLUTION RINGS

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Abstract

In this paper we study Armendariz property for $*$-rings. We introduce the class of $*$-Armendariz $*$-rings, which contains reduced $*$-rings, and its properties are studied. We prove that each $*$-Armendariz $*$-ring is $*$-Abelian. Moreover, we show that the property of a $*$-Armendariz $*$-ring R is extended to its polynomial $*$-ring $R[x]$, localization $S^{-1}R$ of R to S, Laurent polynomial $*$-ring $R[x, x^{-1}]$ and from Ore $*$-ring to its classical Quotient Q. Furthermore, we prove that for a $*$-Armendariz $*$-ring R; R is $*$-Baer if and only if $R[x]$ (resp., $R[[x]]$) is also $*$-Baer. Finally, we show that the property of $*$-ring having quasi-$*$-IFP R can be extendeded to its localization of R to S, Laurent polynomial $*$-ring and polynomial $*$-ring.

1 Introduction

By a ring we always mean an associative ring with identity. A ring R is said to be $*$-*ring* if on R there is defined an involution $*$. $*$-rings are objects of the category of rings with involution with morphisms also preserving involution. Therefore the consistent way of investigating $*$-rings is to study them within this category, as done in a series of papers (for instance [4], [3] and [1]). The

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purpose of this note is to study $*$-Armendariz $*$-rings within its category. The right annihilator of the nonempty set A of R is denoted by $r_R(A)$ and the $*$-right annihilator of A is denoted by $r_{*R}(A) = \{x \in R \mid Ax = Ax^* = 0\}$. If there is no ambiguity, we write $r(A)$ and $r_*(A)$ for $r_R(A)$ and $r_{*R}(A)$, respectively. A self adjoint idempotent element e (that is $e^* = e = e^2$) is called *projection*. A $*$-ring R is said to be *Abelian* ($*$-*Abelian*) if every idempotent (projection) of R is central. We denote the set of all projections of R by $\mathcal{B}_*(R)$. Recall from [4], a nonzero element a of a $*$-ring R is *a $*$-zero divisor* if $ab = 0 = a^*b$ for some nonzero element $b \in R$. Obviously, a $*$-zero divisor element is zero divisor, but the converse is not true [4, Example 3]. A $*$-ring R is said to have *IFP* (*quasi-$*$-IFP*) if for all $a, b \in R, ab = 0$ ($ab = 0 = ab^*$) implies $aRb = 0$ ([11], [1]). R is *reversible* if $ab = 0$ implies $ba = 0$ ([7]).

The study of Armendariz rings which is related to polynomial rings, was initiated by Armendariz [5] and Rege and Chhawchharia [14]. A ring R is called *Armendariz* if whenever polynomials $f(x) = a_0 + a_1x + \ldots + a_mx^m, g(x) = b_0 + b_1x + \ldots + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j. (The converse is obviously true). Recall from [3], an element a of R is said to be *$*$-nilpotent* if $(aa^*)^n = 0$ and $a^m = 0$ for some positive integers n and m. A $*$-ring R is called *reduced* (*$*$-reduced*) if it has no nonzero nilpotent ($*$-nilpotent) elements. Reduced rings are Armendariz by [6, Lemma1]. Following [8], a $*$-ring R is said to be *Baer $*$-ring* if the right annihilator of every nonempty subset of R is generated, as a right ideal, by a projection. In [3], a generalization of Baer $*$-ring is given which is consistent with the category of involution rings that is $*$-Baer $*$-ring. A $*$-ring R is said to be *a $*$-Baer $*$-ring* if the $*$-right annihilator of every nonempty subset A of R is a principal $*$-biideal generated by a projection: that is $r_*(A) = eRe$.
An involution $*$ is called *proper* (resp., *semiproper*) if $aa^* = 0$ (resp., $aRa^* = 0$) implies $a = 0$, for every element $a \in R$. A proper involution is clearly semiproper. Moreover, several examples are included which answers questions that occur naturally in the process of this paper.

Throughout this paper, the integers modulo n will be denoted by $\mathbb{Z}_n$, the field will be denoted by $\mathbb{F}$ and $\mathbb{M}_n(R)$ will denote the full matrix ring of all $n\times n$ matrices over the ring R, while $T_n(R)$ ($T_{nE}(R)$) will denote the $n \times n$ upper triangular matrix ring (with equal diagonal elements) over R. Furthermore, for a commutative ring R, the involution $\diamond$ defined on $\mathbb{T}_{nE}(R)$ for $n > 2$ is given by replacing each entry by its involutive image and fixing the two diagonals considering the diagonal right upper / left lower as symmetric ones and interchanging the symmetric elements about it. For $n = 2$ (trivial extension $\mathbb{T}(R, R)$, the involution $\diamond$ is the adjoint involution.

2 $*$-Armendariz $*$-Rings

In this section, we introduce Armendariz property for $*$-rings. If R is a $*$-ring, then the involution $*$ can naturally be extended to $R[x]$ as:

$$(f(x))^* = (\sum_{i=0}^{m} a_i x^i)^* = \sum_{i=0}^{m} a_i^* x^i \text{ for all } f(x) \in R[x].$$

Definition. A $*$-ring R is called *$*$-Armendariz* if whenever the polynomials $f(x) = a_0 + a_1 x + \ldots + a_m x^m$ and $g(x) = b_0 + b_1 x + \ldots + b_n x^n \in R[x]$ satisfy $f(x)g(x) = f(x)g^*(x) = 0$, then $a_i b_j = 0$ for all i, j (consequently $a_i {b^*}_j = 0$).

Since each Armendariz $*$-ring is clearly $*$-Armendariz and each reduced ring is Armendariz [6, Lemma 1], then we have the following.

Proposition 1. *Each reduced $*$-ring is $*$-Armendariz.*

The converse of the previous proposition is not true as shown by the following example:

Example 1. Consider the $*$-ring $R = \begin{pmatrix} 0 & \mathbb{F} \\ 0 & 0 \end{pmatrix}$, with adjoint involution $*$ defined by: $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^* = \begin{pmatrix} c & -b \\ 0 & a \end{pmatrix}$. R is Armendariz [10, Example 14] and so $*$-Armendariz. Moreover, R is not reduced since the nonzero matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ satisfies $A^2 = 0$.

Example 2. Consider the $*$-ring $R = \begin{pmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix}$, with the adjoint involution $*$. R is not $*$-Armendariz. Indeed, the polynomials $f(x) = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x$, $g(x) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x$, satisfy $f(x)g(x) = f(x)g^*(x) = 0$, while $\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \neq 0$.

For the polynomial $f \in R[x]$ of degree m with $f = \sum_{i=0}^{m} a_i x^i$, let $S_f = \{a_0, a_1, \cdots, a_m\}$.

Corollary 1. *Let R be a reduced $*$-ring and $U \subseteq R[x]$. If $T = U_{f \in U} S_f$ then $\mathrm{r}_{*R[x]}(U) = \mathrm{r}_*(T)[x]$.*

Proof. Let $g = \sum_{j=0}^{n} b_j x^j \in R[x]$ and $Ug = Ug^* = 0$, then $fg = fg^* = 0$ for all $f \in U$ if and only if $a_i b_j = a_i b_j^* = 0$ for all $a_i \in S_f, b_j \in R, 0 \leq j \leq n$, by Proposition 1, which imply

$$\begin{aligned} S_f b_j &= S_f b_j^* = 0 \\ U S_f b_j &= U S_f b_j^* = 0 \\ T b_j &= T b_j^* = 0. \end{aligned}$$

Hence $b_j \in r_*(T)$. The opposite inclusion is clear. $\square$

The question when a $*$-Armendariz $*$-ring is Armendariz has a partial answer in Proposition 2, where we need the following Lemma, which can be easily proved.

Lemma 1. *Let R be a reduced $*$-ring and $f, g \in R[x]$ with $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$. Then $f(gg^*) = f(gg^*)^* = 0$ if and only if $a_i b_j b^*_{k-(i+j)} = 0$ for all $0 \leq i, j \geq k, j \leq k \leq m+n$.*

Proposition 2. *Let R be a $*$-Armendariz $*$-ring with proper involution, then R is Armendariz.*

Proof. Let $f(x)g(x) = 0$ for some $f(x), g(x) \in R[x]$. Then $0 = f(gg^*) = f(gg^*)^*$ implies $a_i c_k = 0$, since R $*$-Armendariz and $c_k = \sum_{j=0}^{k} b_j {b^*}_{k-j}$. Hence $\sum_{i=0}^{k} \sum_{j=0}^{k} a_i (b_j {b^*}_{k-(i+j)}) = 0$ and consequently $a_i b_j {b^*}_{k-(i+j)} = 0$.
Now $(a_i b_j)(a_i b_j)^* = a_i b_j b_j^* a_i^* = 0$. Since $*$ is proper then $a_i b_j = 0$, which means that R is Armendariz. $\square$

One can easily show that the class of $*$-Armendariz $*$-rings is closed under direct sums (with changeless involution) and under taking $*$-subrings.

Proposition 3. *The class of $*$-Armendariz $*$-rings is closed under direct sums and under taking $*$-subrings.*

Using direct proof, we can find $*$-subrings of $\mathbb{T}_{3E}(R)$, which are $*$-Armendariz as follows.

Proposition 4. *Let R be a commutative reduced $*$-ring, then the $\diamond$-ring $\mathbb{T}_{3E}(R)$, with adjoint involution $\diamond$ is $\diamond$-Armendariz.*

Corollary 2. *Let R be a commutative reduced $*$-ring, then the $\diamond$-ring $\mathbb{T}(R, R)$, with adjoint involution $\diamond$ is $\diamond$-Armendariz.*

The reduced condition in Proposition 4 and Corollary 2 is essential according to the following examples:

Example 3. $\mathbb{Z}_4$ is not reduced $*$-ring and the $\diamond$-ring $\mathbb{T}_{3E}(\mathbb{Z}_4)$ is not $\diamond$-Armendariz. Indeed, the polynomial $f(x) = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} x$, satisfies $(f(x))^2 = f(x)f^\diamond(x) = 0$, while $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0$.

Example 4. Again $\mathbb{Z}_8$ is not reduced $*$-ring and the $\diamond$-ring $\mathbb{T}(\mathbb{Z}_8, \mathbb{Z}_8)$ is not $\diamond$-Armendariz. Indeed, the polynomial $f(x) = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix} x$, satisfies $(f(x))^2 = f(x)f^\diamond(x) = 0$, while $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix} \neq 0$.

Based on Proposition 4, one may suspect that $\mathbb{T}_{nE}(R)$ is also $\diamond$-Armendariz for all $n \geq 4$. But the following example discards this possibility.

Example 5. Consider $\mathbb{T}_{4E}(R)$ over a commutative reduced $*$-ring R and let $f(x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x$, $g(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} x$, be polynomials in $\mathbb{T}_{4E}(R)[x]$. Then $f(x)g(x) = f(x)g^{\diamond}(x) = 0$, but $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0$. So $\mathbb{T}_{4E}(R)$ is not $\diamond$-Armendariz. Similarly, for all $n \geq 5$.

The full matrix $\mathbb{M}_n(R)$ over a $*$-ring R with transpose involution is not $*$-Armendariz, for $n \geq 3$, according to the following examples:

Example 6. The $*$-ring $\mathbb{M}_3(R)$ is not $*$-Armendariz. Indeed, the polynomials $f(x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x$, $g(x) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} x$, satisfy $f(x)g(x) = f(x)g^*(x) = 0$, while $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0.$

Example 7. The $*$-ring $\mathbb{M}_4(R)$ is not $*$-Armendariz. Indeed, the polynomials $f(x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x$, $g(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} x$, satisfy $f(x)g(x) = f(x)g^*(x) = 0$, while $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0.$

Using the terminology of Corollary 2, the next example declare that the trivial extension of the trivial extension $\mathbb{T}(R, R)$ (that is; $\mathbb{T}(\mathbb{T}(R, R), \mathbb{T}(R, R))$)

of a commutative reduced $*$-ring is not $\diamond$-Armendariz.

Example 8. Let R be a commutative reduced $*$-ring. Then the $\diamond$-ring $\mathbb{T}(R,R)$ is $\diamond$-Armendariz by Corollary 2 and the $\diamond$-ring $P = \left\{ \begin{pmatrix} A & B \\ 0 & A \end{pmatrix} : A, B \in \mathbb{T}(R,R) \right\}$ is not $\diamond$-Armendariz. Indeed, the polynomial $f(x) = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} x$, satisfy $(f(x))^2 = f(x)f^{\diamond}(x) = 0$, while

$$\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \neq 0.$$

Next, we prove the involutive version of results due to Lee and Zhou ([12]).

Proposition 5. *Every $*$-Armendariz $*$-ring is $*$-Abelian.*

Proof. Let $(ax+b)(a_1x+b_1) = (ax+b)(a_1x+b_1)^* = 0$. If R is not $*$-Armendariz then $-ba_1 = ab_1 \neq 0$ and $-ba_1{}^* = ab_1{}^* \neq 0$; equivalently $br_*(a) \cap ar_*(b) \neq 0$, where $r_*(a)$ (resp., $r_*(b)$) is the a $*$-right annihilator of a (resp., b). Since R is $*$-Armendariz, we have $-ba_1 = ab_1 = 0$ and $-ba_1{}^* = ab_1{}^* = 0$, hence $br_*(a) \cap ar_*(b) = 0$. Let $e_1, e_2 \in R$ be projections and take $b = e_1$ and $a = 1 - e_2$. Noting that $r_*(b) = (1-e_1)R(1-e_1)$ and $r_*(a) = e_2Re_2$, we get $e_1e_2Re_2 \cap (1-e_2)(1-e_1)R(1-e_1) = 0$. Further, suppose that $e_2e_1 = 0$, then $e_1e_2e_2 = e_1e_2 = (1-e_2)(1-e_1)(-e_2)(1-e_1) \in e_1e_2Re_2 \cap (1-e_2)(1-e_1)R(1-e_1) = 0$. Thus for any idempotent $e \in R$ and any element $r \in R$, $x_1 = e+er(1-e), x_2 = e+(1-e)re$ are idempotents satisfy $(1-e)x_1 = 0, x_2(1-e) = 0$ and so $x_1(1-e) = 0, (1-e)x_2 = 0$. Hence $er(1-e) = 0, re(1-e) = 0$ which imply $er = ere, re = ere$. Thus R is Abelian and consequently $*$-Abelian. □

The converse of Proposition 5 is not true according to the following example:

Example 9. By Example 5, the $\diamond$-ring $\mathbb{T}_{4E}(\mathbb{Z}_2)$ is not $\diamond$-Armendariz and the only projections of it are $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ which are central. Hence $\mathbb{T}_{4E}(\mathbb{Z}_2)$ is $\diamond$-Abelian.

A necessary and sufficient conditions for a $*$-ring R to be $*$-Armendariz is now given.

Proposition 6. *For a $*$-ring R, the following statements are equivalent:*

1. R is $*$-Armendariz.

2.eR and $(1-e)R$ are $*$-Armendariz for every projection e of R.

Proof. $1 \Rightarrow 2$ is obvious by Proposition 3.
$2 \Rightarrow 1$. Let $f(x)g(x) = f(x)g^*(x) = 0$ with $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j$
$\in R[x]$, then $ef(x)g(x) = ef(x)eg(x) = ef(x)g^*(x) = ef(x)eg^*(x) = 0$ and $(1-e)f(x)g(x) = (1-e)f(x)(1-e)g(x) = (1-e)f(x)g^*(x) = (1-e)f(x)(1-e)g^*(x) = 0$, since e is central. By assumption, we have $ea_ib_j = 0$ and $(1-e)a_ib_j = 0$ for all $0 \leq i \leq m, 0 \leq j \leq n$. Hence $a_ib_j = ea_ib_j + (1-e)a_ib_j = 0$ and R is $*$-Armendariz. □

In the end of this section, we summarize our main results as follows:

$$\begin{array}{ccccc} \textit{Reduced} & \Rightarrow & \textit{Armendariz} & \Rightarrow & \textit{Abelian} \\ & & \Downarrow & & \Downarrow \\ & & *-\textit{Armendariz} & \Rightarrow & *-\textit{Abelian} \end{array}$$

3 Extensions of $*$-Armendariz $*$-rings

In this section, we generalize the property of $*$-Armendariz to some know extensions; namely the polynomial $*$-ring, the Laurent polynomial $*$-ring, the localization of R to S and from Ore $*$-ring to its classical Quotient.

Theorem 1. *A $*$-ring R is $*$-Armendariz if and only if $R[x]$ is $*$-Armendariz.*

Proof. Let R be a $*$-Armendariz $*$-ring and $f(y)g(y) = f(y)g^*(y) = 0$ with $f(y) = f_0 + f_1 y + \cdots + f_m y^m, g(y) = g_0 + g_1 y + \cdots + g_n y^n \in R[x][y]$ with $f_i, g_j \in R[x]$. Let $t =$ deg $f_0 +$ deg $f_1 + \cdots +$ deg$f_m +$ deg $g_0 +$ deg $g_1 + \cdots +$ deg g_n where the degree is as polynomials in x and the degree of the zero polynomials is taken to be zero. Then $f(x^t) = f_0 + f_1 x^t + \cdots + f_m x^{tm}, g(x^t) = g_0 x^t + g_1 x^t + \cdots + g_n x^{tn} \in R[x]$ and the set of coefficients of the f_i's (resp., g_j's) equals the set of coefficients of the $f(x^t)$ (resp., $g(x^t)$). Since $f(y)g(y) = f(y)g^*(y) = 0$ and x commutes with elements of R, $f(x^t)g(x^t) = f(x^t)g^*(x^t) = 0$. Since R is $*$-Armendariz, each coefficients of f_i annihilates each coefficients of g_i. Thus $f_ig_j = 0$. The sufficient condition is clear by Proposition 3. □

Let R be a $*$-ring and S be a multiplicatively closed subset of R consisting of nonzero central regular elements, then the localization of R to S is the $*$-ring $S^{-1}R = \{u^{-1}a|\ u \in S, a \in R\}$, with involution $*$ defined as:

$$(u^{-1}a)^* = u^{*-1}a^*.$$

Proposition 7. *A $*$-ring R is $*$-Armendariz if and only if $S^{-1}R$ is $*$-Armendariz.*

Proof. By Proposition 3, it suffices to prove the necessary condition.
Let R be a $*$-Armendariz $*$-ring and $F(x)G(x) = F(x)G^*(x) = 0$ with $F(x) = \sum_{i=0}^{m} \alpha_i x^i, G(x) = \sum_{j=0}^{n} \beta_j x^j \in S^{-1}R[x]$, where $\alpha_i = u^{-1}a_i, \beta_j = v^{-1}b_j$, and $a_i, b_j \in R$, $u, v \in S$. Hence

$$\begin{aligned}
F(x)G(x) &= (u^{-1}a_0 + u^{-1}a_1x + \cdots + u^{-1}a_mx^m)(v^{-1}b_0 \\
&\qquad + v^{-1}b_1x + \cdots + v^{-1}b_nx^n) \\
&= u^{-1}v^{-1}a_0b_0 + u^{-1}v^{-1}(a_0b_1 + a_1b_0)x + \cdots \\
&\qquad + u^{-1}v^{-1}(a_0b_n + \cdots + a_mb_0)x^{m+n} \\
&= (vu)^{-1}(a_0b_0 + (a_0b_1 + a_1b_0)x + \cdots \\
&\qquad + (a_0b_n + \cdots + a_mb_0)x^{m+n}) \\
&= (vu)^{-1}f(x)g(x) = 0, \\
F(x)G^*(x) &= (u^{-1}a_0 + u^{-1}a_1x + \cdots + u^{-1}a_mx^m)(v^{-1^*}b_0^* \\
&\qquad + v^{-1^*}b_1^*x + \cdots + v^{-1^*}b_n^*x^n) \\
&= u^{-1}v^{*^{-1}}a_0b_0^* + u^{-1}v^{*^{-1}}(a_0b_1^* + a_1b_0^*)x + \cdots \\
&\qquad + u^{-1}v^{*^{-1}}(a_0b_n^* + \cdots + a_mb_0^*)x^{m+n} \\
&= (v^*u)^{-1}(a_0b_0^* + (a_0b_1^* + a_1b_0^*)x + \cdots \\
&\qquad + (a_0b_n^* + \cdots + a_mb_0^*)x^{m+n}) \\
&= (v^*u)^{-1}f(x)g^*(x) = 0.
\end{aligned}$$

since S is contained in the center of R, so $f(x)g(x) = f(x)g^*(x) = 0$. By hypothesis $a_ib_j = 0$ which implies $\alpha_i\beta_j = (vu)^{-1}a_ib_j = 0$. Therefore $S^{-1}R$ is $*$-Armendariz. □

From Proposition 7, the following results are straightforward.

Corollary 3. *If R is an Armendariz $*$-ring, then $S^{-1}R$ is $*$-Armendariz.*

Corollary 4. *If $S^{-1}R$ is an Armendariz $*$-ring, then R is $*$-Armendariz.*

The $*$-ring of Laurent polynomials in x, with coefficients in a $*$-ring R, consists of all formal sum $f(x) = \sum_{i=k}^{m} a_ix^i$ with obvious addition and multiplication, where $a_i \in R$ and k, m are (possibly negative) integers and with involution $*$ defined as $f^*(x) = \sum_{i=k}^{m} a_i^*x^i$. We denote this ring as usual by $R[x, x^{-1}]$.

Corollary 5. *For a $*$-ring R, $R[x]$ $*$-Armendariz if and only if $R[x, x^{-1}]$ $*$-Armendariz.*

Proof. The sufficient condition is obvious by Proposition 3. Clearly $S = \{1, x, x^2, \cdots\}$ is a multiplicatively closed subset of $R[x]$. Since $R[x, x^{-1}] = S^{-1}R[x]$, it follows that $R[x, x^{-1}]$ is $*$-Armendariz by Proposition 7. □

Recall that a ring R is called right Ore if given $a, b \in R$ with b regular there exist $a_1, b_1 \in R$ with b_1 regular such that $ab_1 = ba_1$. Left Ore is defined similarly and R is *Ore ring* if it is both right and left Ore. For $*$ rings, right Ore implies left Ore and vice versa. It is a known fact that R is Ore if and only if its classical quotient ring Q of R exists and for $*$-rings, $*$ can be extended to Q by $(a^{-1}b)^* = b^*(a^*)^{-1}$ (see[13, Lemma 4]).

Theorem 2. *Let R be an Ore $*$-ring and Q be its classical quotient $*$-ring, then R is $*$-Armendariz if and only if Q is $*$-Armendariz.*

Proof. The sufficiency is clear by Proposition 3 while the necessity is similar to that of [11, Theorem 12]. □

From [11, Theorem 12] and Theorem 2, we have the following.

Corollary 6. *If R is an Armendariz $*$-ring, then Q is $*$-Armendariz.*

Corollary 7. *If Q is an Armendariz $*$-ring, then R is $*$-Armendariz.*

4 Polynomials on $*$-Baer $*$-rings

In this section, we show that the polynomial $*$-ring of $*$-Baer $*$-ring R is $*$-Baer if R is $*$-Armendarize and example is given to show that this condition is not superfluous. Other relative results are also given.

By a similar proof to [10, Lemma 8] or [2, Proposition 11], we have the following.

Lemma 2. *For a $*$-Abelian $*$-ring R. If $e \in \mathcal{B}_*(R[x])$ (resp., $e \in \mathcal{B}_*(R[[x]])$), then $e \in \mathcal{B}_*(R)$.*

As a consequence, we have the following Corollary, from Proposition 5.

Corollary 8. *For a $*$-Armendariz $*$-ring R, if e is a projection in $R[x]$ or $R[[x]]$, then e is a projection in R.*

Proposition 8. *Let R be a $*$-Armendariz $*$-ring, then R is a $*$-Baer $*$-ring if and only if $R[x]$ (resp., $R[[x]]$) is a $*$-Baer $*$-ring.*

Proof. Assume that R is $*$-Baer. Let A be a nonempty subset of $R[x]$ and B be the set of all coefficients of elements of A, then B is a nonempty subset of R and so $r_*(B) = eRe$ for some projection $e \in R$. Since $e \in r_{*R[x]}(A)$ we get $eR[x]e \subseteq r_{*R[x]}(A)$. Now let $g = b_0 + b_1x + \cdots + b_mx^m \in r_{*R[x]}(A)$, then $b_0, b_1, \cdots, b_m \in r_*(B) = eRe$, since R is $*$-Armendariz. Hence there exists $c_0, c_1, \cdots, c_m \in R$ such that $g = ec_0e + ec_1ex + \cdots + ec_mex^m = e(c_0 + c_1x + \cdots + c_mx^m)e \in eR[x]e$ and $R[x]$ is $*$-Baer.
For sufficiency, we prove the result for $R[x]$. Let $R[x]$ be $*$-Baer and D be a subset of R. Since $R[x]$ is $*$-Baer, then there exists a projection $e(x) = e \in R$, by Corollary 8, such that $r_{*R[x]}(D) = eR[x]e$. Hence $r_{*R}(D) = eRe$, since $r_{*R}(D) \subseteq r_{*R[x]}(D) = eR[x]e$. □

Since each reduced $*$-ring is $*$-Armendariz, we have:

Corollary 9. *Let R be a reduced $*$-ring, then R is $*$-Baer if and only if $R[x]$ (resp., $R[[x]]$) is $*$-Baer.*

The next examples shows that the conditions of $*$-Armendariz and reduced in Proposition 8 and Corollary 9, respectively, are essential.

Example 10. By Example 6, the full matrix $*$-ring $\mathbb{M}_3(\mathbb{Z}_3)$, with transpose involution, is not $*$-Armendariz and from [9, Example 2.1] and [3], $\mathbb{M}_n(\mathbb{Z}_3)$ is a $*$-Baer $*$-ring. Moreover, $\mathbb{M}_3(\mathbb{Z}_3)[x]$ is not $*$-Baer, since $r_*(\begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} x)$ cannot be generated by a projection.

Example 11. $\mathbb{M}_2(\mathbb{Z}_3)$ is not reduced $*$-ring and from [9, Example 2.1] and [3], $\mathbb{M}_n(\mathbb{Z}_3)$, with transpose involution, is a $*$-Baer $*$-ring. Moreover, $\mathbb{M}_2(\mathbb{Z}_3)[x]$ is not $*$-Baer, since $r_*(\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x)$ cannot be generated by a projection.

Because each $*$-Baer $*$-ring is $*$-reduced, from Proposition 8, we have the following.

Corollary 10. *For a $*$-Armendariz $*$-ring R, we have the following:*

1. *If R is a $*$- Baer $*$-ring, then $R[x]$ is $*$-reduced.*

2. *If R is a $*$-Baer $*$-ring, then $R[[x]]$ is $*$-reduced.*

5 Some extensions for $*$-rings having quasi-$*$-IFP

In this section, we generalize the property of having quasi-$*$-IFP to some know extensions; namely the localization of R to S, the Laurent polynomial $*$-ring and the polynomial $*$-ring.

By a similar proof to Proposition 7 and using [1, Proposition 2.6], we get analogous result for $*$-rings having quasi $*$-IFP.

Proposition 9. *The $*$-ring R has quasi-$*$-IFP if and only if $S^{-1}R$ has quasi-$*$-IFP.*

Corollary 11. *For a $*$-ring R, $R[x]$ has quasi-$*$-IFP if and only if $R[x, x^{-1}]$ has quasi-$*$-IFP.*

Proof. By [1, Proposition 2.6], it suffices to prove necessity which can be done as the proof of Corollary 5 using Proposition 9. □

Since each $*$-ring having $*$-IFP has quasi-$*$-IFP, from Proposition 9, we have the following relative results.

Corollary 12. *If R has IFP, then $S^{-1}R$ has quasi-$*$-IFP.*

Corollary 13. *If $S^{-1}R$ has IFP, then R has quasi-$*$-IFP.*

Now, we show that the polynomial $*$-ring of a $*$-ring R having quasi-$*$-IFP has quasi-$*$-IFP if R is $*$-Armendariz.

Proposition 10. *For a $*$-ring R, if $R[x]$ has quasi-$*$-IFP, then so is R. The converse holds when R is $*$-Armendariz.*

Proof. Let $R[x]$ have quasi-$*$-IFP, then R has also quasi-$*$-IFP, by [1, Proposition 2.6].
Conversely, let $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) = f(x)g^*(x) = 0$. Since R is $*$-Armendariz, $a_i b_j = 0 = a_i b_j^*$ for each i, j. But R has quasi-$*$-IFP, hence $a_i c_k b_j = 0$ for each i, j and k. It follows that$f(x)h(x)g(x) = 0$ such that $h(x) = \sum_{k=0}^{l} c_k x^k \in R[x]$ and so $R[x]$ has quasi-$*$-IFP. □

From Proposition 10, we have:

Corollary 14. *If R is a reduced $*$-ring and has quasi-$*$-IFP, then $R[x]$ has quasi-$*$-IFP.*

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