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ON THE WEAK AND PARETO QUASI-EQUILIBRIUM PROBLEMS AND THEIR APPLICATIONS

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Abstract

In this paper, we apply a version of Kakutani's fixed point theorem to study weak and Pareto quasi-equilibrium problems. Some sufficient conditions on the existence of solutions of weak and Pareto quasi-equilibrium problems with multivalued mappings are shown. As applications, we give several results on the existence of solutions to vector quasivariational inequalities problems and vector Pareto quasi-saddle problems.

1 Introduction

Let D be a nonempty subset in a real topological vector space X and $f : D \times D \to \mathbb{R}$ be a function such that $f(x, x) = 0$, for all $x \in D$. The problem of finding

$$\bar{x} \in D, \text{ such that } f(\bar{x}, x) \geq 0, \text{ for all } x \in D,$$

is call a scalar equilibrium problem. This problem generalizes many well-known problems in the optimization theory such as variational inequalities, fixed point problems, complementarity problems, saddle point problems, minimax problems (see [2], [5], [8], [10], [13]).

Key words: Quasi-equilibrium problems, quasivariational inequalities problems, quasi-saddle problems, upper and lower C-convex, upper and lower C-quasiconvex-like multivalued mappings, upper and lower C- continuous multivalued mappings, C-pseudomonotone and C-strong pseudomonotone multivalued mappings.
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Now, let X, Y and Z be Hausdorff locally convex topological vector spaces, let $D \subset X, K \subset Z$ be nonempty subsets and let $C \subset Y$ be a cone. We denote $l(C) = C \cap (-C)$. If $l(C) = \{0\}, C$ is said to be pointed. In this paper, we assume that C is a convex closed pointed cone in Y. Given the following multivalued mappings

$$S : D \times K \to 2^D,$$

$$T : D \times K \to 2^K,$$

$$F : K \times D \times D \to 2^Y,$$

we consider the following quasi-equilibrium problems:

(PQEP), Pareto quasi-equilibrium problem: Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\bar{x} \in S(\bar{x}, \bar{y}),$$

$$\bar{y} \in T(\bar{x}, \bar{y}),$$

$$F(\bar{y}, \bar{x}, x) \nsubseteq -C \backslash \{0\}, \text{ for all } x \in S(\bar{x}, \bar{y}).$$

(WQEP), Weak quasi-equilibrium problem: Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\bar{x} \in S(\bar{x}, \bar{y}),$$

$$\bar{y} \in T(\bar{x}, \bar{y}),$$

$$F(\bar{y}, \bar{x}, x) \nsubseteq \text{-int}(C), \text{ for all } x \in S(\bar{x}, \bar{y}).$$

The above problems are natural generalizations of the above scalar equilibrium problem (see [3], [7], [12]). The purpose of this paper is to prove some new results on the existence of solutions to weak and Pareto quasi-equilibrium problems.

2 Preliminaries

Throughout this paper, X, Y and Z we denote real Hausdorff locally convex topological vector spaces. The space of real numbers is denoted by $\mathbb{R}$. Given a subset $D \subset X$, we consider a multivalued mapping $F : D \to 2^Y$. The definition domain and the graph of F are denoted by

$$\text{dom}F = \{x \in D : F(x) \neq \emptyset\},$$

$$\text{Gr}(F) = \{(x, y) \in D \times Y : y \in F(x)\},$$

respectively. We recall that F is said to be a closed mapping if the graph $Gr(F)$ of F is a closed subset in the product space $X \times Y$ and it is said to be a compact mapping if the closure $\overline{F(D)}$ of its range $F(D)$ is a compact set in Y. A multivalued mapping $F : D \to 2^Y$ is said to be upper(lower) semicontinuous

in $\bar{x} \in D$ if for each open set V containing $F(\bar{x})$ (respectively, $F(\bar{x}) \cap V \neq \emptyset$) there exists an open set U of $\bar{x}$ such that $F(x) \subseteq V$ (respectively, $F(x) \cap V \neq \emptyset$) for all $x \in U$.
Now, let Y be a Hausdorff locally convex topological vector space with a cone C. Firstly, we recall the following definitions which will be used in the main results.

Definition 2.1. Let $F : D \to 2^Y$ be a multivalued mapping.

(i) F is said to be upper (lower) C–continuous in $\bar{x} \in$ dom F if for any neighborhood V of the origin in Y there is a neighborhood U of $\bar{x}$ such that:

$$F(x) \subseteq F(\bar{x}) + V + C$$

$$(F(\bar{x}) \subseteq F(x) + V - C, \text{ respectively})$$

holds for all $x \in U \cap \mathrm{dom}F$.

(ii) If F is upper C–continuous and lower C–continuous in $\bar{x}$ simultaneously, we say that it is C–continuous in $\bar{x}$.

(iii) If F is upper, lower,…, C–continuous in any point of dom$F$, we say that it is upper, lower,…, C–continuous on D.

(iv) In the case $C = \{0\}$, a trivial one in Y, we shall only say that F is upper, lower continuous instead of upper, lower 0-continuous. And, F is continuous if it is upper and lower continuous simultaneously.

Definition 2.2. Let F be a multivalued mapping from D to 2^Y. We say that:

(i) F is upper (lower) C-convex on D if for any $x_1, x_2 \in D, t \in [0, 1]$, we have:

$$tF(x_1) + (1-t)F(x_2) \subseteq F(tx_1 + (1-t)x_2) + C$$

$$(\text{respectively}, F(tx_1 + (1-t)x_2) \subseteq tF(x_1) + (1-t)F(x_2) - C).$$

(ii) F is upper (lower) C-quasiconvex-like on D if for any $x_1, x_2 \in D, \alpha \in [0, 1]$, either

$$F(x_1) \subseteq F(\alpha x_1 + (1-\alpha)x_2) + C$$

or,

$$F(x_2) \subseteq F(\alpha x_1 + (1-\alpha)x_2) + C$$

$$(\text{respectively, either } F(\alpha x_1 + (1-\alpha)x_2) \subseteq F(x_1) - C$$

or,

$$F(\alpha x_1 + (1-\alpha)x_2) \subseteq F(x_2) - C)$$

holds.

In [6], Ferro has some examples to show that there is a upper (lower) C-convex multivalued mapping which is not upper (lower) C-quasiconvex-like and conversely, there is also a upper (lower) C-quasiconvex-like multivalued mapping which is not upper (lower) C-convex.

Definition 2.3. Let F be a multivalued mapping from D to 2^Y. We say that:

(i) F is upper (lower) C-hemicontinuous if for any $x, y \in D$, the following implication holds: $F(\alpha x + (1-\alpha)y) \cap C \neq \emptyset$, for all $\alpha \in (0,1)$ implies that $F(y) \cap C(y) \neq \emptyset$ (respectivelly, $F(\alpha x + (1-\alpha)y) \not\subseteq -intC$, for all $\alpha \in (0,1)$ implies that $F(y) \not\subseteq -intC(y)$).

(ii) A multivalued mapping $F : D \longrightarrow 2^Y$ is said to be upper (lower) hemicontinuous if for any $x, y \in D$, the multivalued mapping $f : [0,1] \longrightarrow 2^Y$ defined by $f(\alpha) = F(\alpha x + (1-\alpha)y)$ is upper (respectively, lower) semicontinuous.

Proposition 2.4. *(See [5]) Assume that $F : D \to 2^Y$ is a upper hemicontinuous with nonempty compact values. Then F is upper C-hemicontinuous.*

Definition 2.5. Let $F : D \times D \longrightarrow 2^Y$ be a multivalued mapping. We say that:

(i) F is C- pseudomonotone if for any $x, y \in D$

$$F(y,x) \not\subseteq -int(C) \Longrightarrow F(x,y) \subseteq -C.$$

(ii)F is C- strong pseudomonotone if for any $x, y \in D$

$$F(y,x) \not\subseteq -C\backslash\{0\} \Longrightarrow F(x,y) \subseteq -C.$$

Remark 2.6. If $Y = \mathbb{R}, C = \mathbb{R}_+$ and F is a single-valued mapping then the strongly C- pseudomonotonicity and C- pseudomonotonicity of F become definition for pseudomonotonicity of F in [11].

Example 2.7. Let $D = \mathbb{R}, Y = \mathbb{R}^2, C = \{(t_1; t_2) : t_1 \geq 0, t_2 \in \mathbb{R}\}$ and $F(x,y) = \{(x-y; 0)\}$. Then F is C- pseudomonotone and C- strong pseudomonotone.

Definition 2.8. Let $F : D \longrightarrow 2^D$ be a multivalued mapping. We say that F is a KKM mapping if for each $\{x_1, x_2, ..., x_n\} \subseteq D$, one has

$$co\{x_1, x_2, ..., x_n\} \subseteq \bigcup_{i=1}^{n} F(x_i).$$

The proofs of the following lemmas can be found in [5].

Lemma 2.9. *Let $F : D \times D \to 2^Y$ be a multivalued mapping with nonempty values and $F(x,x) \cap C \neq \emptyset$ for any $x \in D$. In addition, assume that*

(i) *For any fixed $x \in D, F(.,x) : D \to 2^Y$ is upper C-hemicontinuous;*

(ii) *F is C-strong pseudomonotone;*

(iii) *For any fixed $x \in D$, $F(x,.) : D \to 2^Y$ is lower C-convex (or, lower C-quasiconvex-like).*

Then, for any $y \in D$, the following are equivalent.

1) $F(y,x) \not\subseteq -C\backslash\{0\}$, *for all* $x \in D$;

2) $F(x,y) \subseteq -C$, *for all* $x \in D$.

Lemma 2.10. *Let $F : D \times D \to 2^Y$ be a multivalued mapping with nonempty values and $F(x, x) \not\subseteq -intC$ for any $x \in D$. In addition, assume that*

(i) *For any fixed $x \in D, F(., x) : D \to 2^Y$ is lower C-hemicontinuous;*

(ii) *F is C- pseudomonotone;*

(iii) *For any fixed $x \in D$, $F(x, .) : D \to 2^Y$ is lower C-convex.*

Then, for any $y \in D$, the followings are equivalent:

1) $F(y, x) \not\subseteq -intC$, *for all* $x \in D$;

2) $F(x, y) \subseteq -C$, *for all* $x \in D$.

In the proof of the main results in Section 3, we need the following theorems.

Theorem 2.11. *(See [4]) Assume that X is a topological vector space, $D \subseteq X$ is nonempty convex compact and $F : D \to 2^D$ is a KKM mapping with closed values. Then, we have*

$$\bigcap_{x \in D} F(x) \neq \emptyset.$$

Theorem 2.12. *(Kakutani fixed point theorem, see [1]) Let D be a nonempty convex compact subset and $F : D \to 2^D$ be a multivalued mapping closed with nonempty convex values. Then there exists $\bar{x} \in D$ such that $\bar{x} \in F(\bar{x})$.*

3 Main Results

Throughout this section, unless otherwise specify, by X, Y and Z we denote Hausdorff locally convex topological vector spaces. Let $D \subset X, K \subset Z$ be nonempty subsets, C is a convex closed pointed cone in Y . Given the following multivalued mappings

$$S : D \times K \longrightarrow 2^D,$$

$$T : D \times K \longrightarrow 2^K,$$

$$F : K \times D \times D \longrightarrow 2^Y,$$

we prove that following theorem:

Theorem 3.1. *Let D and K be nonempty convex compact subsets of Hausdorff locally convex topological vector space X and Z, respectively. Assume that the multivalued mapping F with nonempty values and $F(y, x, x) \cap C \neq \emptyset$, for all $(x, y) \in D \times K$. In addition, assume that:*

(i) *S is a continuous multivalued mapping with nonempty convex closed values;*

(ii) *T is a upper semicontinuous multivalued mapping with nonempty convex closed values;*

(iii) *For each $y \in K$, $F(y, ., .) : D \times D \to 2^Y$ is C-strong pseudomonotone;*

(iv) *For any fixed $(x, y) \in D \times K$, the multivalued mapping $F(y, x, .) : D \to 2^Y$ is lower C-convex (or, lower C-quasiconvex-like);*

(v) *F is lower C-continuous and for any fixed $(y,z) \in K \times D$, $F(y,.,z)$ is upper C-hemicontinuous.*
Then there exists $(\bar{x}, \bar{y}) \in D \times K$ such that $\bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y})$ and

$$F(\bar{y}, \bar{x}, x) \nsubseteq -C\backslash\{0\}, \text{ for all } x \in S(\bar{x}, \bar{y}).$$

Proof. We define the multivalued mapping $M : D \times K \to 2^D$ by

$$M(x,y) = \{x' \in S(x,y) : F(y,z,x') \subseteq -C, \text{ for all } z \in S(x,y)\}.$$

For each $(x,y) \in D \times K$, we will show that $M(x,y)$ is nonempty set. Indeed, for each $(x,y) \in D \times K$, we define the multivalued mapping $Q_{xy} : S(x,y) \to 2^{S(x,y)}$ by

$$Q_{xy}(z) = \{x' \in S(x,y) : F(y,z,x') \subseteq -C\}.$$

Let $\{x'_\alpha\}$ be a net in $Q_{xy}(z)$, $x'_\alpha \to x'$. We have $x'_\alpha \in S(x,y)$ and $F(y,z,x'_\alpha) \subseteq -C$. Since $S(x,y)$ is a closed set, so $x' \in S(x,y)$. On the other hand, F is lower C-continuous, for any neighborhood V of the origin in Y, there exists an index α_0 such that

$$F(y,z,x') \subseteq F(y,z,x'_\alpha) - C + V, \text{ for all } \alpha \geq \alpha_0.$$

This implies that

$$F(y,z,x') \subseteq -C + V.$$

Since C is closed, we have

$$F(y,z,x') \subseteq -C.$$

Hence $x' \in Q_{xy}(z)$ and $Q_{xy}(z)$ is closed set.

Now we show that Q_{xy} is a KKM type mapping. If not, then there exists $\{x_1, x_2, ..., x_n\} \subseteq S(x,y)$ such that

$$co\{x_1, x_2, ..., x_n\} \nsubseteq \bigcup_{i=1}^{n} Q_{xy}(x_i).$$

Hence there exists $x^* \in co\{x_1, x_2, ..., x_n\}$ and $x^* \notin Q_{xy}(x_i)$, for $i = 1, 2, ..., n$. This implies

$$F(y, x_i, x^*) \nsubseteq -C, \text{ for } i = 1, 2, ..., n.$$

Since $F(y,.,.)$ is C-strong pseudomonotone, we deduce that

$$F(y, x^*, x_i) \subseteq -C\backslash\{0\}, \text{ for } i = 1, 2, ..., n.$$

Since $F(y,x,.)$ is lower C-convex (or, lower C-quasiconvex-like), we imply

$$F(y, x^*, x^*) \subseteq -C\backslash\{0\}.$$

This contradicts with $F(y,x,x) \cap C \neq \emptyset$. Therefore Q_{xy} is a KKM mapping.

By Theorem 2.11, we have $\cap_{z \in S(x,y)} Q_{xy}(z) \neq \emptyset$. Hence, there exists $x' \in S(x,y)$ such that $F(y,z,x') \subseteq -C$, for all $z \in S(x,y)$. Thus, $M(x,y) \neq \emptyset$.

We show that $M(x,y)$ is convex set. Indeed, let $x'_1, x'_2 \in M(x,y)$ and $t \in [0,1]$, we have from the convexity of $S(x,y)$, $tx'_1 + (1-t)x'_2 \in S(x,y)$ and

$$F(y,z,x'_1) \subseteq -C,$$

$$F(y,z,x'_2) \subseteq -C, \text{ for all } z \in S(x,y).$$

Since $F(y,x,.)$ is lower C-convex (or, lower C-quasiconvex-like), we conclude

$$F(y,z,tx'_1 + (1-t)x'_2) \subseteq -C, \text{ for all } z \in S(x,y).$$

This shows $tx'_1 + (1-t)x'_2 \in M(x,y)$ and $M(x,y)$ is a convex set.

Further, we claim that M is a closed multivalued mapping. Let $x_\alpha \to x, y_\alpha \to y, x'_\alpha \in M(x_\alpha, y_\alpha), x'_\alpha \to x'$. We show that $x' \in M(x,y)$. Indeed, since $x'_\alpha \in S(x_\alpha, y_\alpha)$ and the upper semicontinuity of S with closed values, $x' \in S(x,y)$. For $x'_\alpha \in M(x_\alpha, y_\alpha)$, we have

$$F(y_\alpha, z, x'_\alpha) \subseteq -C, \text{ for all } z \in S(x_\alpha, y_\alpha).$$

For each $z \in S(x,y)$, by the lower semicontinuity of S, there exists $z_\alpha \in S(x_\alpha, y_\alpha)$ such that $z_\alpha \to z$. We have

$$F(y_\alpha, z_\alpha, x'_\alpha) \subseteq -C.$$

Since F is lower C-continuous, for any neighborhood V of the origin in Y, there exists an index α_0 such that

$$F(y,z,x') \subseteq F(y_\alpha, z_\alpha, x'_\alpha) - C + V, \text{ for all } \alpha \geq \alpha_0.$$

This implies that

$$F(y,z,x') \subseteq -C + V.$$

Since C is closed, we have

$$F(y,z,x') \subseteq -C.$$

This means that $x' \in M(x,y)$ and M is a closed multivalued mapping.

Lastly, we define the multivalued mapping $P : D \times K \longrightarrow 2^{D \times K}$ by

$$P(x,y) = M(x,y) \times T(x,y)$$

We can easily verify that P is a closed multivalued mapping with nonempty convex values. Moreover, since $D \times K$ is a compact set, we have that P is also a upper semicontinuous multivalued mapping with nonempty convex closed values. Applying the fixed point theorem of Kakutani type, there exists $(\bar{x},\bar{y}) \in P(\bar{x},\bar{y})$. This implies $\bar{x} \in S(\bar{x},\bar{y}), \bar{y} \in T(\bar{x},\bar{y})$ and

$$F(\bar{y},x,\bar{x}) \subseteq -C, \text{ for all } x \in S(\bar{x},\bar{y}).$$

We use Lemma 2.9 with D replaced by $S(\bar{x},\bar{y})$, we have $\bar{x} \in S(\bar{x},\bar{y}), \bar{y} \in T(\bar{x},\bar{y})$ and

$$F(\bar{y},\bar{x},x) \not\subseteq -C\backslash\{0\}, \text{ for all } x \in S(\bar{x},\bar{y}).$$

The proof of the corollary is complete. □

By using Lemma 2.10 and the proof is similar as the one of Theorem 3.1, we obtain the following result.

Theorem 3.2. *Let D and K be nonempty convex compact subsets of Hausdorff locally convex topological vector space X and Z, respectively. Assume that the multivalued mapping F with nonempty values and $F(y,x,x) \not\subseteq -int(C)$, for all $(x,y) \in D \times K$. In addition, assume that:*

(i) *S is a continuous multivalued mapping with nonempty convex closed values;*

(ii) *T is a upper semicontinuous multivalued mapping with nonempty convex closed values;*

(iii) *For any fixed $y \in K$, $F(y,.,.) : D \times D \to 2^Y$ is C-pseudomonotone;*

(iv) *For any fixed $(x,y) \in D \times K$, $F(y,x,.) : D \to 2^Y$ is lower C-convex;*

(v) *F is lower C-continuous and for any fixed $(y,z) \in K \times D$, $F(y,.,z)$ is lower C-hemicontinuous.*

Then there exists $(\bar{x},\bar{y}) \in D \times K$ such that $\bar{x} \in S(\bar{x},\bar{y}), \bar{y} \in T(\bar{x},\bar{y})$ and

$$F(\bar{y},\bar{x},x) \not\subseteq -int(C), \text{ for all } x \in S(\bar{x},\bar{y}).$$

Remark 3.3. The assumption (v) in Theorem 3.1 and Theorem 3.2 can be replaced by the following condition:

(v') The set $\{(x,y,z) \in D\times K\times D : F(y,x,z) \subseteq -C\}$ is closed in $D\times K\times D$.

4 System of quasi-equilibrium problems

Now, given D,K,C,S,T as above and $G : K\times D\times D \longrightarrow 2^Y, H : D\times K\times K \longrightarrow 2^Y$ are multivalued mappings with nonempty values. We consider the following problems:

(SPQEP), System of Pareto quasi-equilibrium problems: Find $(\bar{x},\bar{y}) \in D \times K$ such that

$$\bar{x} \in S(\bar{x},\bar{y}), \bar{y} \in T(\bar{x},\bar{y})$$

and

$$G(\bar{y},\bar{x},x) \not\subseteq -C \setminus \{0\}, \text{for all } x \in S(\bar{x},\bar{y}),$$

$$H(\bar{x}, \bar{y}, y) \not\subseteq -C \setminus \{0\}, \text{for all } y \in T(\bar{x}, \bar{y}).$$

(SWQEP), System of weak quasi-equilibrium problems: Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y})$$

and

$$G(\bar{y}, \bar{x}, x) \not\subseteq -int(C), \text{for all } x \in S(\bar{x}, \bar{y}),$$
$$H(\bar{x}, \bar{y}, y) \not\subseteq -int(C), \text{for all } y \in T(\bar{x}, \bar{y}).$$

Theorem 4.1. *Let D and K be nonempty convex compact subsets of Hausdorff locally convex topological vector space X and Z, respectively. Assume that the multivalued mappings G, H with nonempty values and $G(y, x, x) \cap C \neq \emptyset, H(x, y, y) \cap C \neq \emptyset$ for all $(x, y) \in D \times K$. The following conditions are sufficient for (SPQEP) to have a solution:*

(i) *S, T are continuous multivalued mappings with nonempty convex closed values;*

(ii) *$G(y, ., .), H(x, ., .)$ are C-strong pseudomonotone, for any fixed $(x, y) \in D \times K$;*

(iii) *$G(y, x, .) : D \to 2^Y, H(x, y, .) : K \to 2^Y$ are lower C-convex (or, lower C-quasiconvex), for every $(x, y) \in D \times K$ fixed;*

(iv) *$G(y, ., x), H(x, ., y)$ are upper C-hemicontinuous, for any fixed $(x, y) \in D \times K$;*

(v) *G, H are lower C-continuous.*

Proof. We define the multivalued mappings $M_1 : D \times K \to 2^D, M_2 : D \times K \to 2^K$ by

$$M_1(x, y) = \{x' \in S(x, y) : G(y, z, x') \subseteq -C, \quad \text{for all} \quad z \in S(x, y)\}.$$

$$M_2(x, y) = \{y' \in T(x, y) : H(x, t, y') \subseteq -C, \quad \text{for all} \quad t \in T(x, y)\}.$$

Then we can early prove that M_1, M_2 are closed mappings with nonempty convex values. Now, we define the multivalued mapping $M : D \times K \to 2^{D \times K}$ by

$$M(x, y) = M_1(x, y) \times M_2(x, y).$$

Then M is closed mapping with nonempty convex. Applying theorem fixed point Kakutani type, there exists $(\bar{x}, \bar{y}) \in D \times K$ such that $(\bar{x}, \bar{y}) \in M(\bar{x}, \bar{y})$. This implies, $\bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y})$ and

$$G(\bar{y}, x, \bar{x}) \subseteq -C, \text{for all } x \in S(\bar{x}, \bar{y}),$$

$$H(\bar{x}, y, \bar{y}) \subseteq -C, \text{for all } y \in T(\bar{x}, \bar{y}).$$

Since $G(y, ., .), H(x, ., .)$ are C-strong pseudomonotone, we have

$$G(\bar{y}, \bar{x}, x) \not\subseteq -C \setminus \{0\}, \text{for all } x \in S(\bar{x}, \bar{y}),$$

$$H(\bar{x}, \bar{y}, y) \not\subseteq -C \setminus \{0\}, \text{for all } y \in T(\bar{x}, \bar{y}).$$

The proof of the theorem is complete. □

By exploiting the similar arguments used in the proof of Theorem 4.1, we obtain the following result.

Theorem 4.2. *Assume that D and K are nonempty convex compact subsets of Hausdorff locally convex topological vector space X and Z, respectively. Let F, G be set-valued maps with nonempty values and $G(y, x, x) \not\subseteq -int(C), H(x, y, y) \not\subseteq -int(C)$ for all $(x, y) \in D \times K$. The following conditions are sufficient for (SWQEP) to have a solution:*

(i) *S, T are continuous multivalued mappings with nonempty convex closed values;*

(ii) *For any fixed $(x, y) \in D \times K$, $G(y, ., .), H(x, ., .)$ are C-pseudomonotone;*

(iii) *For every $(x, y) \in D \times K$ fixed, the multivalued mappings $G(y, x, .) : D \to 2^Y, H(x, y, .) : K \to 2^Y$ are lower C-convex;*

(iv) *For any fixed $(x, y) \in D \times K$, $G(y, ., x), H(x, ., y)$ are lower C-hemicontinuous;*

(v) *G, H are lower C-continuous.*

5 Applications to vector quasi-variational inequalities problems

In this section, we apply the obtained results in Section 3 to vector quasi-variational inequalities problems with multivalued mappings. Let $L(X, Y)$ be the set of all continuous linear mappings from X into Y and $f(x)$ denote the value of f at x where $f \in L(X, Y)$, $x \in X$. Let $D \subset X, K \subset Z$ be nonempty subsets, let $\phi : D \longrightarrow Y$ be a single valued mapping and $S : D \times K \longrightarrow 2^D, T : D \times K \longrightarrow 2^K, G : D \times K \longrightarrow 2^{L(X,Y)}$ be multivalued mappings. In addition, assume that C is a pointed convex closed cone in Y. We consider the following problem:

Vector weak quasi-variational inequalities problem: Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y})$$

and

$$G(\bar{x}, \bar{y})(x - \bar{x}) + \phi(x) - \phi(\bar{x}) \not\subseteq -int(C), \text{for all } x \in S(\bar{x}, \bar{y}).$$

Vector Pareto quasi-variational inequalities problem: Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y})$$

and

$$G(\bar{x}, \bar{y})(x - \bar{x}) + \phi(x) - \phi(\bar{x}) \not\subseteq -C \setminus \{0\}, \text{for all } x \in S(\bar{x}, \bar{y}).$$

Definition 5.1. *Let $F : D \to 2^{L(X,Y)}$ be a multivalued mapping. We say that:*

(i) F is $C-$pseudomonotone with respect to ϕ if for any given $x, z \in D$

$$F(x)(x-z)+\phi(z)-\phi(x) \not\subseteq -int(C) \Longrightarrow F(z)(z-x)+\phi(x)-\phi(z) \subseteq -C.$$

(ii) F is $C-$strong pseudomonotone with respect to ϕ if for any given $x, z \in D$

$$F(x)(x-z)+\phi(z)-\phi(x) \not\subseteq -C \setminus \{0\} \Longrightarrow F(z)(z-x)+\phi(x)-\phi(z) \subseteq -C.$$

Corollary 5.2. *Let D, K, S, T be the same as in Theorem 3.1. In addition, assume that:*

(i) *The mapping ϕ is lower C-convex;*

(ii) *For any fixed $y \in K$, the mapping $G(.,y) : D \to 2^{L(X,Y)}$ is $C-$ strong pseudomonotone with respect to ϕ;*

(iii) *For any fixed $(y,z) \in K \times D$, the mapping $x \longmapsto G(x,y)(z-x)+\phi(z)-\phi(x)$ is upper C-hemicontinuous;*

(iv) *The set $\{(x,y,z) \in D \times K \times D : G(x,y)(z-x)+\phi(z)-\phi(x) \subseteq -C\}$ is closed in $D \times K \times D$.*

Then the above vector Pareto quasi-variational inequalities problem has a solution.

Proof. The proof of this corollary follows immidiately from Theorem 3.1 and Remark 3.3 by taking $F(y,x,z) = G(x,y)(z-x)+\phi(z)-\phi(x)$. □

Corollary 5.3. *Let D, K, S, T be the same as in Theorem 3.2. In addition, assume that:*

(i) *The mapping ϕ is lower C-convex;*

(ii) *For any fixed $y \in K$, the mapping $G(.,y) : D \to 2^{L(X,Y)}$ is $C-$pseudomonotone with respect to ϕ;*

(iii) *For any fixed $(y,z) \in K \times D$, the mapping $x \longmapsto G(x,y)(z-x)+\phi(z)-\phi(x)$ is lower C-hemicontinuous;*

(iv) *The set $\{(x,y,z) \in D \times K \times D : G(x,y)(z-x)+\phi(z)-\phi(x) \subseteq -C\}$ is closed in $D \times K \times D$.*

Then the above vector weak quasi-variational inequalities problem has a solution.

Proof. The proof of this corollary follows immidiately from Theorem 3.2 and Remark 3.3 by taking $F(y,x,z) = G(x,y)(z-x)+\phi(z)-\phi(x)$. □

6 Applications to vector Pareto quasi-saddle problems

Let $D \subset X, K \subset Z$ be nonempty subsets, let $f : D \times K \longrightarrow Y$ be a single valued mapping and $S : D \times K \longrightarrow 2^D, T : D \times K \longrightarrow 2^K$ be multivalued

mappings. In addition, assume that C is a pointed convex closed cone in Y satisfying : $Y = C + (-C)$. We consider the following problem.
Vector Pareto quasi-saddle problem: Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y})$$

and

$$f(x, \bar{y}) \notin f(\bar{x}, \bar{y}) - C \setminus \{0\}, \text{for all } x \in S(\bar{x}, \bar{y}),$$

$$f(\bar{x}, \bar{y}) \notin f(\bar{x}, y) - C \setminus \{0\}, \text{for all } y \in T(\bar{x}, \bar{y}).$$

Using the results obtained in the previous section, we establish a existence result for solutions of this problem.

Corollary 6.1. *Let D, K, S, T be the same as in Theorem 4.1. In addition, assume that:*

(i) *The mapping f is $(-C)$-continuous and C-continuous;*

(ii) *For any fixed $(x, y) \in D \times K$, the mapping $f(., y) : D \longrightarrow Y$ is C-concave (or, C-quasiconcave-like) and $f(x, .) : K \longrightarrow Y$ is C- convex(or, C-quasiconvex-like).*

Then the above vector Pareto quasi-saddle problem has a solution.

Proof. We define the single valued mappings $G : K \times D \times D \longrightarrow Y, H : D \times K \times K \longrightarrow Y$ by

$$G(y, x, z) = f(z, y) - f(x, y), H(x, y, t) = f(x, y) - f(x, t).$$

Then, the vector Pareto quasi-saddle problem becomes to find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y})$$

and

$$G(\bar{y}, \bar{x}, x) \nsubseteq -C \setminus \{0\}, \text{for all } x \in S(\bar{x}, \bar{y}),$$

$$H(\bar{x}, \bar{y}, y) \nsubseteq -C \setminus \{0\}, \text{for all } y \in T(\bar{x}, \bar{y}).$$

First of all, we show that $G(y, ., z)$ is upper C-hemicontinuous. Indeed, assume that

$$G(y, \alpha x_1 + (1 - \alpha)x_2, z) \cap C \neq \emptyset, \text{ for all } \alpha \in (0, 1).$$

This implies

$$[f(z, y) - f(\alpha x_1 + (1 - \alpha)x_2, y)] \cap C \neq \emptyset, \text{ for all } \alpha \in (0, 1).$$

By f is $(-C)$-continuous, for an arbitrary neighborhood V of the origin in Y, we have

$$f(\alpha x_1 + (1 - \alpha)x_2, y) \in f(x_2, y) + V + C.$$

This implies

$$[f(z,y) - f(x_2,y) - V - C] \cap C \neq \emptyset.$$

Hence, we have

$$[f(z,y) - f(x_2,y) + V] \cap C \neq \emptyset.$$

This gives

$$[f(z,y) - f(x_2,y)] \cap C \neq \emptyset.$$

Hence, $G(y,.,z)$ is upper C-hemicontinuous. By the similar arguments used in the above proof, we conclude that $H(x,.,t)$ is upper C-hemicontinuous.

Now, we show that $G(y,.,.)$ is strong C-pseudomonotone. Suppose $G(y,x,z) \not\subseteq -C \setminus \{0\}$ namely, $f(z,y) - f(x,y) \notin -C \setminus \{0\}$ and hence $f(x,y) - f(z,y) \notin C \setminus \{0\}$. Since $Y = C + (-C)$, we conclude that $f(x,y) - f(z,y) \in -C$. Therefore $G(y,z,x) \subseteq -C$. Hence $G(y,.,.)$ is strong C-pseudomonotone. By the similar arguments used in the above proof, we conclude that $H(x,.,.)$ is strong C-pseudomonotone.

Next, we show that for any fixed $(x,y) \in D \times K$, $G(y,x,.)$ is lower C-convex (or, lower C-quasiconvex-like). Let $z_1, z_2 \in D$ and $\alpha \in [0,1]$, if $f(.,y)$ is C-concave, then we have

$G(y,x,\alpha z_1 + (1-\alpha)z_2) = f(\alpha z_1 + (1-\alpha)z_2, y) - f(x,y) \in \alpha f(z_1,y) + (1-\alpha)f(z_2,y) - f(x,y) - C = \alpha G(y,x,z_1) + (1-\alpha)G(y,x,z_2) - C$. Hence $G(y,x,.)$ is lower C-convex. If $f(x,.)$ is C-quasiconcave-like, we also conclude that $G(y,x,.)$ is lower C-quasiconvex-like. By the similar arguments used in the above proof, we conclude that $H(x,y,.)$ is lower C-convex(or, lower C-quasiconvex-like).

We claim that G is lower C-continuous. Indeed, let $(y_0,x_0,z_0) \in K \times D \times D$. Since f is $(-C)$-continuous and C-continuous, for an arbitrary neighborhood V of the origin in Y there exists neighborhoods $U_{x_0}, U_{y_0}, U_{z_0}$ of x_0, y_0, z_0, such that

$$f(z_0,y_0) \in f(z,y) + V - C, \text{ for all } (z,y) \in (U_{z_0}, U_{y_0}).$$

$$f(x_0,y_0) \in f(x,y) + V + C, \text{ for all } (x,y) \in (U_{x_0}, U_{y_0}).$$

Then, we have

$$f(z_0,y_0) - f(x_0,y_0) \in f(z,y) - f(x,y) + V - C, \text{ for all } (x,y,z) \in (U_{x_0}, U_{y_0}, U_{z_0}).$$

This mean that

$$G(y_0,x_0,z_0) \subseteq G(y,x,z) + V - C, \text{ for all } (x,y,z) \in (U_{x_0}, U_{y_0}, U_{z_0}).$$

Hence, G is lower C-continuous. By the similar arguments used in the above proof, we conclude that H is lower C-continuous.

Applying Theorem 4.1, there exists $(\bar{x},\bar{y}) \in D \times K$ such that

$$\bar{x} \in S(\bar{x},\bar{y}), \bar{y} \in T(\bar{x},\bar{y})$$

and

$$G(\bar{y}, \bar{x}, x) \not\subseteq -C \setminus \{0\}, \text{for all } x \in S(\bar{x}, \bar{y}),$$

$$H(\bar{x}, \bar{y}, y) \not\subseteq -C \setminus \{0\}, \text{for all } y \in T(\bar{x}, \bar{y}).$$

This mean that $\bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y})$ and

$$f(x, \bar{y}) \notin f(\bar{x}, \bar{y}) - C \setminus \{0\}, \text{for all } x \in S(\bar{x}, \bar{y}),$$

$$f(\bar{x}, \bar{y}) \notin f(\bar{x}, y) - C \setminus \{0\}, \text{for all } y \in T(\bar{x}, \bar{y}).$$

The proof of the theorem is complete. □ When $Y = \mathbb{R}, C = \mathbb{R}_+$, we have the following corollary.

Corollary 6.2. *Let D, K, S, T be the same as in Corollary 4.1. In addition, assume that:*

(i) *The mapping $f : D \times K \to \mathbb{R}$ is continuous;*

(ii) *For any fixed $(x, y) \in D \times K$, the mapping $f(., y) : D \longrightarrow \mathbb{R}$ is concave (or, quasiconcave) and $f(x, .) : K \longrightarrow \mathbb{R}$ is convex(or, quasiconvex).*

Then there exists $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y})$$

and

$$\max_{x \in S(\bar{x}, \bar{y})} \min_{y \in T(\bar{x}, \bar{y})} f(x, y) = \min_{y \in T(\bar{x}, \bar{y})} \max_{x \in S(\bar{x}, \bar{y})} f(x, y).$$

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