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SOME RESULTS ON SLICES AND ENTIRE GRAPHS IN CERTAIN WEIGHTED WARPED PRODUCTS

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Abstract

We study the area-minimizing property of slices in the weighted warped product manifold $(\mathbb{R}^+ \times_f \mathbb{R}^n, e^{-\varphi})$, assuming that the density function $e^{-\varphi}$ and the warping function f satisfy some additional conditions. Based on a calibration argument, a slice $\{t_0\} \times \mathbb{G}^n$ is proved weighted area-minimizing in the class of all entire graphs satisfying a volume balance condition and some Bernstein type theorems in $\mathbb{R}^+ \times_f \mathbb{G}^n$ and $\mathbb{G}^+ \times_f \mathbb{G}^n$, when f is constant, are obtained.

1 Introduction

Recently, the study of weighted minimal submanifolds, and in particular weighted minimal hypersurfaces had attracted many researchers (see, for instance, [2], [4], [5], [7]). A weighted manifold (also called a manifold with density) is a Riemannian manifold endowed with a positive function $e^{-\varphi}$, called the density, used to weight both volume and perimeter elements. The weighted area of a hypersurface Σ in an $(n+1)$-dimensional weighted manifold is $\mathrm{Area}_\varphi(\Sigma) = \int_\Sigma e^{-\varphi}\, dA$ and the weighted volume of a region Ω is $\mathrm{Vol}_\varphi(\Omega) = \int_\Omega e^{-\varphi}\, dV$, where dA and dV are the n-dimensional Riemannian area and $(n+1)$-dimensional Riemannian volume elements, respectively. A typical example of such manifolds is Gauss space $\mathbb{G}^{n+1}$, $\mathbb{R}^{n+1}$ with Gaussian

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density $(2\pi)^{-\frac{n+1}{2}}e^{-\frac{r^2}{2}}$, which is appeared in probability and statistics. The hypersurface Σ in $\mathbb{R}^{n+1}$ is said to be weighted minimal or φ-minimal if

$$H_\varphi(\Sigma) := H(\Sigma) + \frac{1}{n}\langle \nabla\varphi, N\rangle = 0,$$

where $H(\Sigma)$ and N are the classical mean curvature and the unit normal vector field of Σ, respectively. $H_\varphi(\Sigma)$ is called the weighted mean curvature of Σ.

A theme widely approached in recent years is problems concerning to hypersurfaces in a warped product manifold of the type $\mathbb{R}^+ \times_f M$, where $\mathbb{R}^+ = [0, +\infty)$, (M, g) is an n-dimensional Riemannian manifold and f is a positive smooth function defined on $\mathbb{R}^+$ (see [8]). Note that with these ingredients, the product manifold $\mathbb{R}^+ \times_f M$ is endowed with the Riemannian metric

$$\bar{g} = \pi^*_{\mathbb{R}^+}(dt^2) + f(\pi_{\mathbb{R}^+})^2\pi^*_M(g),$$

where $\pi_{\mathbb{R}^+}$ and π_M denote the projections onto $\mathbb{R}^+$ and M, respectively.

In $\mathbb{R}^n$, let P be a part of a slice, viewed as a graph over a domain D and let Σ be a graph of a function u over D. It is clear that

$$\mathrm{Area}(\Sigma) = \int_D \sqrt{1 + |\nabla u|^2}\, dA \geq \int_D dA = \mathrm{Area}(P).$$

However, in general, the above inequality doesn't always hold if the ambient space is a weighted manifold. For instance, consider $\mathbb{R}^2$ with radial density $e^{-\frac{1}{2}(x^2+y^2)}$. Let R be a positive real number, $P = \{(x, 0) \in \mathbb{R}^2 : \ -R \leq x \leq R\}$ and Σ be the half circle defined by $x^2 + y^2 = R^2$, $y \geq 0$. The weighted length of P, $L_\varphi(P)$, and the weighted length of Σ, $L_\varphi(\Sigma)$, are

$$L_\varphi(P) = \int_{-R}^{R} e^{-\frac{1}{2}x^2}\, dx,$$

and

$$L_\varphi(\Sigma) = \int_0^\pi e^{-\frac{1}{2}R^2} R\, dt = e^{-\frac{1}{2}R^2} R\pi.$$

A simple computation shows that $\sqrt{2\pi(1 - e^{-\frac{1}{2}R^2})} \leq L_\varphi(P) \leq \sqrt{\pi(1 - e^{-R^2})}$. When $R = 2$, we have $L_\varphi(P) \geq L_\varphi(\Sigma)$.

As another example, we consider $\mathbb{R}^2$ with density e^y. Let

$$P = \left\{\left(x, -\ln\cos\frac{\pi}{3}\right) \in \mathbb{R}^2 : \ -\frac{\pi}{3} \leq x \leq \frac{\pi}{3}\right\}$$

and Σ be the graph of function $y = -\ln\cos x, x \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$. It's not hard to check that $L_\varphi(P) \geq L_\varphi(\Sigma)$.

Hence, the area-minimizing property of slices in weighted warped product manifolds is not a trivial matter. In this paper, using the same method as in [2] we prove that if $(\log f)''(t) \le 0$, then the slice is weighted area-minimizing under a volume balance condition. In particular, when f is constant we get some Bernstein type theorems in $\mathbb{R}^+ \times_f \mathbb{G}^n$ and $\mathbb{G}^+ \times_f \mathbb{G}^n$.

2 Preliminaries

Consider the warped product $\mathbb{R}^+ \times_f \mathbb{R}^n$ with density $e^{-\varphi}$, where $\varphi = \varphi(t, x)$. Let $u \in C^2(\mathbb{R}^n)$, and $\Sigma = \{(u(x), x) : x \in \mathbb{R}^n\}$ be the entire graph defined by u. A unit normal vector field of Σ is

$$N = \left(\frac{f(u)}{\sqrt{f(u)^2 + |Du|^2}}, -\frac{1}{f(u)\sqrt{f(u)^2 + |Du|^2}} Du \right),$$

where Du is the gradient of u in $\mathbb{R}^n$, and $|Du|^2 = \langle Du, Du \rangle$. The curvature function (relative to N) is $H = \dfrac{1}{n}\text{trace}(A)$, where A is the shape operator. A direct computation gives (see [8, Section 5])

$$nH(u) = \text{div}\left(\frac{Du}{f(u)\sqrt{f^2 + |Du|^2}} \right) - \frac{f'(u)}{\sqrt{f(u)^2 + |Du|^2}} \left(n - \frac{|Du|^2}{f(u)^2} \right).$$

Thus,

$$\begin{aligned} nH_\varphi(u) = \frac{1}{f(u)}\text{div}\left(\frac{Du}{\sqrt{f(u)^2 + |Du|^2}} \right) &- \frac{nf'(u)}{\sqrt{f(u)^2 + |Du|^2}} + \frac{f(u)}{\sqrt{f(u)^2 + |Du|^2}}\varphi_t \\ &- \frac{1}{f(u)\sqrt{f(u)^2 + |Du|^2}} \langle Du, D\varphi \rangle. \end{aligned}$$

It is easy to see that the mean curvature as well as the weighted mean curvature of slice are constants

$$H(t_0) := H(t_0, x) = -(\log f)'(t_0),$$

and

$$H_\varphi(t_0) := H_\varphi(t_0, x) = -(\log f)'(t_0) + \varphi_t(t_0, x).$$

Furthermore, if $\varphi = \varphi(x)$, $x \in \mathbb{R}^n$ (i.e., the weighted function $e^{-\varphi}$ does not depend on the parameter $t \in \mathbb{R}^+$), $H_\varphi(t_0) = -(\log f)'(t_0)$.

Let Σ and N as above. Consider the smooth extension of N by the translation along t-axis, also denoted by N and the n-differential form defined by

$$\phi(t, x) = f(t)^n \omega(x),$$

where $\omega(X_1,...,X_n)=\det(X_1,...,X_n,N)$, $X_i, i=1,2,...,n$, are smooth vector fields on Σ. It is clear that $f(t)^n|\omega(X_1,...,X_n)|\le 1$, for all orthonormal vector fields $X_i, i=1,2,...,n$ and $f(t)^n|\omega(X_1,...,X_n)|=1$ if and only if $X_1,...,X_n$ are tangent to Σ. Therefore, $\phi(t,x)$ represents the weighted volume element of Σ in $(\mathbb{R}^+\times_f\mathbb{R}^n, e^{-\varphi})$. We have

$$\mathrm{div}N=-nH-\frac{f'}{\sqrt{f^2+|Du|^2}}\left(n-\frac{|Du|^2}{f^2}\right)+\frac{f'|Du|^2}{(f^2+|Du|^2)^{\frac{3}{2}}}.$$

Note that $d\omega=\mathrm{div}(N)\,dV_{\mathbb{R}^+\times\mathbb{R}^n}$, thus

$$\begin{aligned}d\phi&=d(f^n\omega)=\mathrm{div}(f^nN)\,dV_{\mathbb{R}^+\times\mathbb{R}^n}=f^n\mathrm{div}N\,dV_{\mathbb{R}^+\times\mathbb{R}^n}+nf^{n-1}f'\langle\partial_t,N\rangle\,dV_{\mathbb{R}^+\times\mathbb{R}^n}\\&=\mathrm{div}N\,dV_{\mathbb{R}^+\times_f\mathbb{R}^n}+n\frac{f'}{f}\langle\partial_t,N\rangle\,dV_{\mathbb{R}^+\times_f\mathbb{R}^n}\\&=\left(-nH+\frac{f'|Du|^2}{f^2\sqrt{f^2+|Du|^2}}+\frac{f'|Du|^2}{(f^2+|Du|^2)^{\frac{3}{2}}}\right)dV_{\mathbb{R}^+\times_f\mathbb{R}^n}.\end{aligned}$$

Since

$$\begin{aligned}d(e^{-\varphi}\phi)&=d(e^{-\varphi}f^n\omega)=e^{-\varphi}f^n\mathrm{div}N\,dV_{\mathbb{R}^+\times\mathbb{R}^n}+\langle\nabla(e^{-\varphi}f^n),N\rangle\,dV_{\mathbb{R}^+\times\mathbb{R}^n}\\&=e^{-\varphi}d\phi-e^{-\varphi}f^n\langle\nabla\varphi,N\rangle\,dV_{\mathbb{R}^+\times\mathbb{R}^n}\\&=e^{-\varphi}\left[-nH+\frac{f'|Du|^2}{f^2\sqrt{f^2+|Du|^2}}+\frac{f'|Du|^2}{(f^2+|Du|^2)^{\frac{3}{2}}}-\langle\nabla\varphi,N\rangle\right]dV_{\mathbb{R}^+\times_f\mathbb{R}^n}\\&=e^{-\varphi}\left[-nH_\varphi+\frac{f'|Du|^2}{f^2\sqrt{f^2+|Du|^2}}+\frac{f'|Du|^2}{(f^2+|Du|^2)^{\frac{3}{2}}}\right]dV_{\mathbb{R}^+\times_f\mathbb{R}^n},\end{aligned}$$

we have

$$d_\varphi\phi=e^{\varphi}d(e^{-\varphi}\phi)=\left(-nH_\varphi+\frac{f'|Du|^2}{f^2\sqrt{f^2+|Du|^2}}+\frac{f'|Du|^2}{(f^2+|Du|^2)^{\frac{3}{2}}}\right)dV_{\mathbb{R}^+\times_f\mathbb{R}^n}.$$

When Σ is a slice, $d_\varphi\phi=-nH_\varphi\,dV_{\mathbb{R}^+\times_f\mathbb{R}^n}$.

3 The results

3.1 The results on slices

Consider $\mathbb{R}^+\times_f\mathbb{R}^n$ with density $e^{-\varphi}$, $\varphi=\varphi(t,x)$. Suppose that D is a domain in $\mathbb{R}^n$ such that $\overline{D}$, the closure of D, is compact. Let $P_D=\{t_0\}\times D$ and Σ_D be the graph of a function $t=u(x)$, $x\in D$, such that P_D and Σ_D have the same boundary, i.e., $\partial P_D=\partial\Sigma_D$. Let $E_1=\{(t,x)\in\mathbb{R}^+\times D: t\le u(x)\}$ and $E_2=\{(t,x)\in\mathbb{R}^+\times D: t\le t_0\}$. The following theorem shows that P_D has least weighted area in the class of hypersurfaces with the same boundary.

Theorem 3.1. *If* $\mathrm{Vol}_\varphi(E_1) = \mathrm{Vol}_\varphi(E_2)$ *and* $(\log f)''(t) \leq 0$, *then* $\mathrm{Area}_\varphi(P_D) \leq \mathrm{Area}_\varphi(\Sigma_D)$.

Proof. Denote by ϕ the volume form of $\mathbb{R}^n$. By Stokes' Theorem and the suitable orientations for objects (see Figure 1), we get

$$\begin{aligned}\mathrm{Area}_\varphi(D) - \mathrm{Area}_\varphi(\Sigma_D) &\leq \int_D e^{-\varphi}\phi - \int_{\Sigma_D} e^{-\varphi}\phi = \int_{D-\Sigma_D} e^{-\varphi}\phi \\ &= \int_{E_1} e^{-\varphi} d_\varphi\phi = \int_{E_1\setminus E_2} e^{-\varphi} d_\varphi\phi + \int_{E_1\cap E_2} e^{-\varphi} d_\varphi\phi,\end{aligned}$$

$$\begin{aligned}\mathrm{Area}_\varphi(P_D) - \mathrm{Area}_\varphi(D) &\leq \int_{P_D} e^{-\varphi}\phi - \int_D e^{-\varphi}\phi = \int_{P_D-D} e^{-\varphi}\phi \\ &= -\int_{E_2} e^{-\varphi} d_\varphi\phi = -\int_{E_2\setminus E_2} e^{-\varphi} d_\varphi\phi - \int_{E_1\cap E_2} e^{-\varphi} d_\varphi\phi.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{Area}_\varphi(P_D) - \mathrm{Area}_\varphi(\Sigma_D) &\leq \int_{E_1\setminus E_2} e^{-\varphi} d_\varphi\phi - \int_{E_2\setminus E_2} e^{-\varphi} d_\varphi\phi \\ &= -\int_{E_1\setminus\ E_2} e^{-\varphi} nH_\varphi(t)\, dV + \int_{E_2\setminus E_2} e^{-\varphi} nH_\varphi(t)\, dV.\end{aligned}$$

The condition $(\log f)''(t) \leq 0$ means that H_φ is non-decreasing along t-axis.

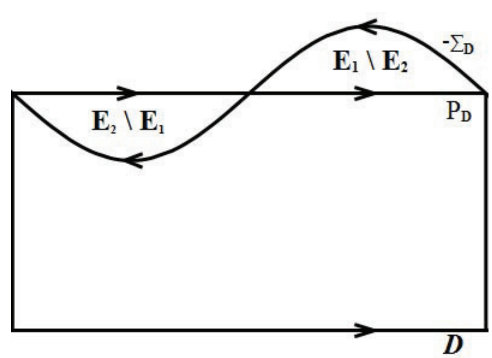

Figure 1: A part of slice and graph have the same boundary

Therefore,

$$H_\varphi(t_0) \leq H_\varphi(t),\ \forall (t,x) \in E_1 \setminus E_2; \quad H_\varphi(t) \leq H_\varphi(t_0),\ \forall (t,x) \in E_2 \setminus E_1.$$

Hence

$$\begin{aligned}\mathrm{Area}_\varphi(P_D) - \mathrm{Area}_\varphi(\Sigma_D) &\le -nH_\varphi(t_0)\left(\int_{E_1\setminus E_2} e^{-\varphi}\, dV - \int_{E_2\setminus E_1} e^{-\varphi}\, dV\right)\\ &= -nH_\varphi(t_0)(Vol_\varphi(E_1\setminus E_2) - \mathrm{Vol}_\varphi(E_2\setminus E_1)) = 0,\end{aligned}$$

because $\mathrm{Vol}_\varphi(E_1) = \mathrm{Vol}_\varphi(E_2)$. Thus, $\mathrm{Area}_\varphi(P_D) \le \mathrm{Area}_\varphi(\Sigma_D)$. □

In the case of $\mathbb{R}^n$ is the Gauss space G^n, consider $\mathbb{R}^+ \times_f \mathbb{G}^n$, i.e., $\mathbb{R}^+ \times_f \mathbb{R}^n$ with density $e^{-\varphi} = (2\pi)^{-n/2}e^{-\frac{|x|^2}{2}}$. In this space, slices are proved to be global weighted area-minimizing.

Theorem 3.2. *If* $(\log f)''(t) \le 0$, *then a slice is weighted area-minimizing in the class of all entire graphs satisfying* $\mathrm{Vol}_\varphi(E_1) = \mathrm{Vol}_\varphi(E_2)$.

Proof. Let P be the slice $\{t_0\}\times\mathbb{G}^n$ and Σ be the graph of a function $t = u(x)$ over $\mathbb{G}^n$. Let S_R^{n-1} be the $(n-1)$-sphere with center O and radius R in $\mathbb{G}^n$ and $C_R = \mathbb{R}\times S_R^{n-1}$ be the n-dimensional cylinder. Let $E_1 = \{(t,x)\in\mathbb{R}^+\times\mathbb{G}^n : t\le u(x)\}$ and $E_2 = \{(t,x)\in\mathbb{R}^+\times\mathbb{G}^n : t\le t_0\}$. Let $A = E_1\setminus E_2 \ \cup\ E_2\setminus E_1$. The parts of P, Σ, E_1, and E_2, bounded by C_R, are denoted by P_R, Σ_R, E_{1_R}, and E_{2_R}, respectively.

Denote by ϕ the volume form of $\mathbb{G}^n$. Let R be large enough such that C_R meets both $E_1\setminus E_2$ and $E_2\setminus E_1$ (see Figure 2). In a similar way to the proof of Theorem 3.1, we have

$$\begin{aligned}\mathrm{Area}_\varphi(\mathbb{G}_R^n) - \mathrm{Area}_\varphi(\Sigma_R) + \int_{C_R\cap E_1} e^{-\varphi}\phi &\le \int_{\mathbb{G}_R^n} e^{-\varphi}\phi - \int_{\Sigma_R} e^{-\varphi}\phi + \int_{C_R\cap E_1} e^{-\varphi}\phi\\ &= \int_{E_{1_R}} e^{-\varphi} d_\varphi\phi = \int_{E_{1_R}\setminus E_{2_R}} e^{-\varphi} d_\varphi\phi + \int_{E_{1_R}\cap E_{2_R}} e^{-\varphi} d_\varphi\phi,\end{aligned}$$

$$\begin{aligned}\mathrm{Area}_\varphi(P_R) - \mathrm{Area}_\varphi(\mathbb{G}_R^n) + \int_{C_R\cap E_2} e^{-\varphi}\phi &\le \int_{P_R} e^{-\varphi}\phi - \int_{\mathbb{G}_R^n} e^{-\varphi}\phi + \int_{C_R\cap E_2} e^{-\varphi}\phi\\ &= -\int_{E_{2_R}} e^{-\varphi} d_\varphi\phi = -\int_{E_{2_R}\setminus E_{1_R}} e^{-\varphi} d_\varphi\phi - \int_{E_{2_R}\cap E_{1_R}} e^{-\varphi} d_\varphi\phi.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{Area}_\varphi(P_R) - \mathrm{Area}_\varphi(\Sigma_R) + \int_{C_R\cap A} e^{-\varphi}\phi &\le \int_{E_{1_R}\setminus E_{2_R}} e^{-\varphi} d_\varphi\phi - \int_{E_{2_R}\setminus E_{2_R}} e^{-\varphi} d_\varphi\phi\\ &= \int_{E_{2_R}\setminus E_{1_R}} e^{-\varphi} nH_\varphi(t)\, dV - \int_{E_{1_R}\setminus E_{2_R}} e^{-\varphi} nH_\varphi(t)\, dV. \quad (3.1)\end{aligned}$$

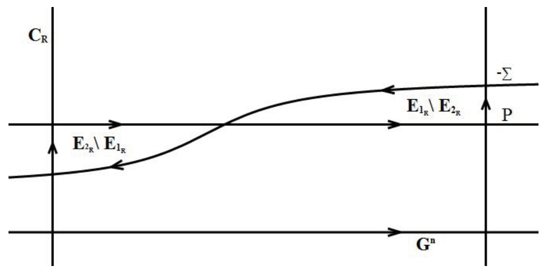

Figure 2: The slice P, entire graph Σ and $\mathbb{G}^n$ in $\mathbb{R}^+ \times_f \mathbb{G}^n$

Since $(\log f)''(t) \leq 0$,

$H_\varphi(t_0) \leq H_\varphi(t), \forall (t,x) \in E_{1_R} \setminus E_{2_R}$ and $H_\varphi(t) \leq H_\varphi(t_0), \forall (t,x) \in E_{2_R} \setminus E_{1_R}$.

Thus,

$$\mathrm{Area}_\varphi(P_R) - \mathrm{Area}_\varphi(\Sigma_R) + \int_{C_R \cap A} e^{-\varphi}\phi \leq nH_\varphi(t_0)\left(\mathrm{Vol}_\varphi(E_{2_R} \setminus E_{1_R}) - \mathrm{Vol}_\varphi(E_{1_R} \setminus E_{2_R})\right). \tag{3.2}$$

Moreover, it is easy to see that $\lim_{R\to\infty} \int_{C_R\cap A} e^{-\varphi}\phi = \lim_{R\to\infty} e^{-cR^2} \int_{C_R\cap A} \phi = 0$.
By the assumption $\mathrm{Vol}_\varphi(E_1) = \mathrm{Vol}_\varphi(E_2)$, we have

$$\lim_{R\to\infty} \mathrm{Vol}_\varphi(E_{1_R} \setminus \ E_{2_R}) = \lim_{R\to\infty} \mathrm{Vol}_\varphi(E_{2_R} \setminus \ E_{1_R}).$$

Hence, taking the limit of both sides of (3.2) as R goes to infinity, we obtain $\mathrm{Area}_\varphi(P) \leq \mathrm{Area}_\varphi(\Sigma)$. □

3.2 Some Bernstein type results

3.2.1 A Bernstein type result in $\mathbb{R}^+ \times_a \mathbb{G}^n$

Consider the weighted warped product manifold $\mathbb{R}^+ \times_a \mathbb{G}^n$ with density $e^{-\varphi} = (2\pi)^{-n/2} e^{-\frac{|x|^2}{2}}$, where a is a positive constant. Let P, Σ, E_1, E_2, A, C_R, P_R, Σ_R, E_{1_R}, E_{2_R} be defined as in the proof of Theorem 3.2. If u is bounded, then $\mathrm{Vol}_\varphi(E_1)$, $\mathrm{Vol}_\varphi(E_2)$ and $\mathrm{Vol}_\varphi(A)$ are finite. Since the weighted mean curvature of Σ on the region A, H_φ, does not change along any vertical line, we get the following results:

Theorem 3.3. *If $H_\varphi(\Sigma)$ and u are bounded and $\mathrm{Vol}_\varphi(E_1) = \mathrm{Vol}_\varphi(E_2)$, then*

$$\mathrm{Area}_\varphi(\Sigma) \leq \mathrm{Area}_\varphi(P) + \frac{1}{2} n(M-m)\,\mathrm{Vol}_\varphi(A),$$

where $m = \inf H_\varphi(\Sigma)$ and $M = \sup H_\varphi(\Sigma)$.

Proof. Denote by ϕ the volume form of Σ. In this case, $d_\varphi \phi = -nH_\varphi \, dV$. Let R be large enough such that C_R meets both $E_1 \setminus E_2$ and $E_2 \setminus E_1$ (see Figure 2). By changing Σ_R and P_R together in (3.1), we have

$$\begin{aligned} \mathrm{Area}_\varphi(\Sigma_R) - \mathrm{Area}_\varphi(P_R) + \int_{C_R \cap A} e^{-\varphi}\phi &\le \int_{{E_1}_R \setminus {E_2}_R} e^{-\varphi} nH_\varphi(\Sigma)\, dV - \int_{{E_2}_R \setminus {E_1}_R} e^{-\varphi} nH_\varphi(\Sigma)\, dV \\ &\le nM \,\mathrm{Vol}_\varphi({E_1}_R \setminus {E_2}_R) - nm \,\mathrm{Vol}_\varphi({E_2}_R \setminus {E_1}_R). \end{aligned} \tag{3.3}$$

By the assumption $\mathrm{Vol}_\varphi(E_1) = \mathrm{Vol}_\varphi(E_2)$, taking the limit of both sides of (3.3) as R goes to infinity, we get $\mathrm{Area}_\varphi(\Sigma) \le \mathrm{Area}_\varphi(P) + \dfrac{1}{2} n(M - m)\,\mathrm{Vol}_\varphi(A)$. □

Corollary 3.4 (Bernstein type theorem in $\mathbb{R}^+ \times_a \mathbb{G}^n$). *A bounded entire constant mean curvature graph must be a slice and therefore, is minimal.*

Proof. Assume that Σ is an entire constant mean curvature graph of a bounded function u. Since $\mathrm{Vol}_\varphi(E_1)$ is finite, there exists a slice P such that $\mathrm{Vol}_\varphi(E_1) = \mathrm{Vol}_\varphi(E_2)$. Because $m = M$, by Theorem 3.3, it follows that $\mathrm{Area}_\varphi(\Sigma) \le \mathrm{Area}_\varphi(P)$. Moreover,

$$\mathrm{Area}_\varphi(\Sigma) = \int_{\mathbb{G}^n} e^{-\varphi} \sqrt{a^4 + a^2|Du|^2}\, dA \ge \int_{\mathbb{G}^n} e^{-\varphi} \sqrt{a^4}\, dA = \mathrm{Area}_\varphi(P).$$

Therefore, $\mathrm{Area}_\varphi(\Sigma) = \mathrm{Area}_\varphi(P)$ and $Du = 0$, i.e., u is constant. It is not hard to see that $\Sigma = P$ and therefore, is minimal. □

3.2.2 A Bernstein type result in $\mathbb{G}^+ \times_a \mathbb{G}^n$

Now, consider the weighted warped product manifold $\mathbb{G}^+ \times_a \mathbb{G}^n$ with density $e^{-\varphi} = (2\pi)^{-(n+1)/2} e^{-\frac{r^2}{2}}$, and let Σ be an entire graph of a function $u(x)$ over $\mathbb{G}^n$, since

$$\begin{aligned} &\langle \nabla\varphi(u(x) + \Delta t, x), N(u(x) + \Delta t, x)\rangle - \langle \nabla\varphi(u(x), x), N(u(x), x)\rangle \\ &= \langle (u(x) + \Delta t, x) - (u(x), x), N\rangle = \langle (\Delta t, 0), N\rangle \ge 0, \quad \text{for} \quad \Delta t \ge 0, \end{aligned}$$

the weighted mean curvature of Σ is increasing along any vertical line. We have

Lemma 3.5.

$$\mathrm{Area}_\varphi(\mathbb{G}^n) \le \mathrm{Area}_\varphi(\Sigma).$$

Proof. Denote by ϕ the volume form of $\mathbb{G}^n$. Replacing C_R by S_R, the n-sphere with center O and radius R, in Subsection 3.2.1. Let R be large enough such that S_R meets Σ (see Figure 3), we get

$$\mathrm{Area}_\varphi(\mathbb{G}^n_R) - \mathrm{Area}_\varphi(\Sigma_R) + \int_{S_R \cap E_1} e^{-\varphi}\phi \le -\int_{{E_1}_R} e^{-\varphi} nH_\varphi(\mathbb{G}^n)\, dV = 0.$$

Therefore, $\mathrm{Area}_\varphi(\mathbb{G}^n) \le \mathrm{Area}_\varphi(\Sigma)$. □

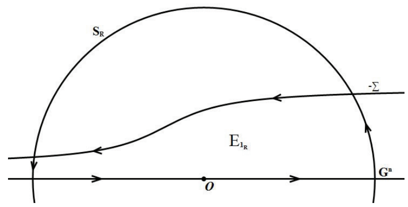

Figure 3: An entire graph Σ and $\mathbb{G}^n$ in $\mathbb{G}^+ \times_a \mathbb{G}^n$

Theorem 3.6 (Bernstein type theorem in $\mathbb{G}^+ \times_a \mathbb{G}^n$). *The only entire weighted minimal graph in $\mathbb{G}^+ \times_a \mathbb{G}^n$ is $\mathbb{G}^n$.*

Proof. Denote by ϕ the volume form of Σ (see Figure 3), we have

$$\operatorname{Area}_\varphi(\Sigma_R) - \operatorname{Area}_\varphi(\mathbb{G}^n_R) + \int_{S_R \cap E_1} e^{-\varphi}\phi \leq \int_{E_{1_R}} e^{-\varphi} n H_\varphi(\Sigma)\, dV = 0. \quad (3.4)$$

Taking the limit of both sides of (3.4) as R goes to infinity, we get

$$\operatorname{Area}_\varphi(\Sigma) \leq \operatorname{Area}_\varphi(\mathbb{G}^n).$$

Hence, it follows from Lemma 3.5 that

$$\operatorname{Area}_\varphi(\Sigma) = \operatorname{Area}_\varphi(\mathbb{G}^n). \quad (3.5)$$

Since $\operatorname{Vol}_\varphi(\mathbb{G}^+ \times_a \mathbb{G}^n)$ is finite, there exists a slice P such that $\operatorname{Vol}_\varphi(E_1) = \operatorname{Vol}_\varphi(E_2)$. Using the similar arguments as in the proof of Theorem 3.3 (see Figure 4), we get

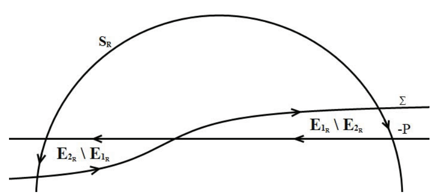

Figure 4: The slice P and entire graph Σ in $\mathbb{G}^+ \times_a \mathbb{G}^n$

$$\operatorname{Area}_\varphi(\Sigma_R) - \operatorname{Area}_\varphi(P_R) + \int_{S_R \cap A} e^{-\varphi}\phi \leq \int_{E_{1_R} \setminus E_{2_R}} e^{-\varphi} n H_\varphi(\Sigma)\, dV - \int_{E_{2_R} \setminus E_{1_R}} e^{-\varphi} n H_\varphi(\Sigma)\, dV = 0,$$

because Σ is a weighted minimal graph. Therefore, $\operatorname{Area}_\varphi(\Sigma) \leq \operatorname{Area}_\varphi(P)$. By Theorem 3.2, it follows that

$$\operatorname{Area}_\varphi(\Sigma) = \operatorname{Area}_\varphi(P). \quad (3.6)$$

Thus, it follows from (3.5) and (3.6) that

$$\mathrm{Area}_{\varphi}(P) = \mathrm{Area}_{\varphi}(\mathbb{G}^n).$$

Hence, $P = \mathbb{G}^n$ and $\mathrm{Vol}_{\varphi}(E_1) = \mathrm{Vol}_{\varphi}(E_2) = 0$, i.e., $\Sigma = \mathbb{G}^n$. □

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