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ON (σ,τ)-$\star$-DERIVATION AND COMMUTATIVITY OF $\star$-PRIME RING

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Abstract

In this paper we study the notion of (σ,τ)-$\star$-derivation and prove the following result: Let $\mathcal{R}$ be a $\star$-prime ring with characteristic different from two and $Z(\mathcal{R})$ be the center of $\mathcal{R}$. If $\mathcal{R}$ admits a non-zero (σ,τ)-$\star$-derivation d of $\mathcal{R}$, with associated automorphisms σ and τ of $\mathcal{R}$, such that σ, τ and d commute with $\star$ satisfying $[d(U), d(U)]_{\sigma,\tau} = \{0\}$, then $\mathcal{R}$ is commutative, where U is an ideal of $\mathcal{R}$ such that $U^\star = U$.

1 Introduction

Throughout, $\mathcal{R}$ will denote an associative ring with center $Z(\mathcal{R})$. An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is said to be a derivation of $\mathcal{R}$ if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in \mathcal{R}$. For a fixed $a \in \mathcal{R}$, the mapping $I_a : \mathcal{R} \to \mathcal{R}$ given by $I_a(x) = [a, x] = ax - xa$ is a derivation which is said to be an inner derivation. Recall that $\mathcal{R}$ is said to be prime if $a\mathcal{R}b = \{0\}$ implies $a = 0$ or $b = 0$. A ring $\mathcal{R}$ is said to be 2-torsion free, if $2x = 0$ implies $x = 0$.

For any two endomorphisms σ and τ of $\mathcal{R}$, we call an additive mapping $d : \mathcal{R} \to \mathcal{R}$ a (σ,τ)-derivation of $\mathcal{R}$ if $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ for all $x, y \in \mathcal{R}$. Of course, a $(1, 1)$-derivation is a derivation on $\mathcal{R}$, where 1 is the

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identity mapping on $\mathcal{R}$. We set $[x,y]_{\sigma,\tau} = x\sigma(y) - \tau(y)x$. In particular $[x,y]_{1,1} = [x,y] = xy - yx$, is the usual Lie product.

An additive mapping $x \mapsto x^\star$ on a ring $\mathcal{R}$ is called an involution if $(x^\star)^\star = x$ and $(xy)^\star = y^\star x^\star$ hold for all $x, y \in \mathcal{R}$. A ring equipped with an involution is called a ring with involution or $\star$-ring. A ring $\mathcal{R}$ equipped with an involution $\star$ is said to be $\star$-prime if $a\mathcal{R}b = a\mathcal{R}b^\star = \{0\}$ (or, equivalently $a\mathcal{R}b = a^\star\mathcal{R}b = \{0\}$) implies $a = 0$ or $b = 0$. It is important to note that, a prime ring is $\star$-prime, but the converse is in general not true. An example due to Shulaing [13] justifies this fact. If $\mathcal{R}^\circ$ denotes the opposite ring of a prime ring $\mathcal{R}$, then $S = \mathcal{R} \times \mathcal{R}^\circ$ equipped with the exchange involution $\star_{ex}$ defined by $\star_{ex}(x,y) = (y,x)$ is $\star_{ex}$-prime, but not a prime ring because of the fact that $(1,0)S(0,1) = 0$. In all that follows, $Sa_\star(\mathcal{R})$ will denote the set of symmetric and skew symmetric elements of $\mathcal{R}$, i.e., $Sa_\star(\mathcal{R}) = \{x \in \mathcal{R} | x^\star = \pm x\}$. An ideal U of $\mathcal{R}$ is said to be a $\star$-ideal of $\mathcal{R}$ if $U^\star = U$. It can also be noted that an ideal of a ring $\mathcal{R}$ may not be $\star$-ideal of $\mathcal{R}$. As an example, let $\mathcal{R} = \mathbb{Z} \times \mathbb{Z}$, and consider an involution $\star$ on $\mathcal{R}$ such that $(a,b)^\star = (b,a)$ for all $(a,b) \in \mathcal{R}$. The subset $U = \mathbb{Z} \times \{0\}$ of $\mathcal{R}$ is an ideal of $\mathcal{R}$ but it is not a $\star$-ideal of $\mathcal{R}$, because $U^\star = \{0\} \times \mathbb{Z} \neq U$.

Let $\mathcal{R}$ be a ring with involution $\star$. An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is said to be a $\star$-derivation if $d(xy) = d(x)y^\star + xd(y)$ holds for all $x, y \in \mathcal{R}$. The concept of $\star$-derivation was introduced by Brešar and Vukman in [8]. In [1], Shakir and Fošner introduced (σ,τ)-$\star$-derivation as follows: Let σ and τ be two endomorphism of $\mathcal{R}$. An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is said to be (σ,τ)-$\star$-derivation if $d(xy) = d(x)\sigma(y^\star) + \tau(x)d(y)$, holds for all $x, y \in \mathcal{R}$. In [8], Brešar and Vukman studied some algebraic properties of $\star$-derivations.

Recently many authors have studied commutativity of prime and semiprime rings with involution admitting suitably constrained derivations (for reference see [2, 12, 16, 20] etc). A lot of work have been done by L. Okhtite and his co-authors on rings with involution (see for reference [17, 18, 19], where further references can be found).

In [15], Lee and Lee proved that if a prime ring of characteristic different from 2 admits a derivation d such that $[d(\mathcal{R}), d(\mathcal{R})] \subseteq Z(\mathcal{R})$, then $\mathcal{R}$ is commutative. On the other hand in [11] for $a \in \mathcal{R}$, Herstein proved that if $[a, d(\mathcal{R})] = \{0\}$, then $a \in Z(\mathcal{R})$. Further in the year 1992, Aydin together with Kaya [7] extended the theorems mentioned above by replacing derivation by (σ,τ)-derivation and in some of those, $\mathcal{R}$ by a non-zero ideal of $\mathcal{R}$. Recently, in [4] we investigated the commutativity of $\star$-prime ring $\mathcal{R}$ equipped with an involution $\star$ admitting a (σ,τ)-derivation d satisfying $[d(U), d(U)]_{\sigma,\tau} = \{0\}$, where U is a nonzero $\star$-ideal of $\mathcal{R}$. In this paper we prove the above mentioned theorem in case of (σ,τ)-$\star$-derivation. In fact, it is shown that if a $\star$-prime ring admits a nonzero (σ,τ)-$\star$-derivation d satisfying $[d(U), d(U)]_{\sigma,\tau} = \{0\}$, then $\mathcal{R}$ is commutative.

2 The Results

In the remaining part of the paper, $\mathcal{R}$ will represent a $\star$-prime ring which admits a nonzero (σ, τ)-$\star$-derivation d with automorphisms σ and τ such that $\star$ commutes with d, σ and τ. We shall use the following relations frequently without specific mention:

$$[xy, z]_{\sigma,\tau} = x[y, z]_{\sigma,\tau} + [x, \tau(z)]y = x[y, \sigma(z)] + [x, z]_{\sigma,\tau}y,$$

$$[x, yz]_{\sigma,\tau} = \tau(y)[x, z]_{\sigma,\tau} + [x, y]_{\sigma,\tau}\sigma(z),$$

and

$$[x, [y, z]]_{\sigma,\tau} + [[x, z]_{\sigma,\tau}, y]_{\sigma,\tau} - [[x, y]_{\sigma,\tau}, z]_{\sigma,\tau} = 0.$$

Remark 2.1. *We find that if $\mathcal{R}$ is a $\star$-prime ring with characteristic different from 2, then $\mathcal{R}$ is a 2-torsion free. In fact, if $2x = 0$ for all $x \in \mathcal{R}$, then $xr(2s) = 0$ for all $r, s \in \mathcal{R}$. But since char $\mathcal{R} \neq 2$, there exists a nonzero $l \in \mathcal{R}$ such that $2l \neq 0$ and hence by the above $x\mathcal{R}2l = \{0\}$. This also gives that $x\mathcal{R}(2l)^\star = \{0\}$ and $\star$-primeness of $\mathcal{R}$ yields that $x = 0$, i.e., $\mathcal{R}$ is 2-torsion free.*

The main result of the present paper states as follows:

Theorem 2.2. Let $\mathcal{R}$ be a $\star$- prime ring with characteristic different from two and σ, τ be automorphisms of $\mathcal{R}$, and U a $\star$-ideal of $\mathcal{R}$. If $\mathcal{R}$ admits a non-zero (σ, τ)-$\star$-derivation $d : \mathcal{R} \to \mathcal{R}$ such that $[d(U), d(U)]_{\sigma,\tau} = \{0\}$, then $\mathcal{R}$ is commutative.

We facilitate our discussion with the following lemmas which are required for developing the proof of our main result.

Since every $\star$-prime ring is semiprime and every $\star$-right ideal is right ideal. Hence Lemma 1.1.5 of [9] can be rewritten in case of $\star$-prime ring as follows:

Lemma 2.3. Let $\mathcal{R}$ be a $\star$-prime ring and U a non-zero $\star$-right ideal of $\mathcal{R}$. Then $Z(U) \subseteq Z(\mathcal{R})$.

Corollary 2.4. Let $\mathcal{R}$ be a $\star$-prime ring and U a non-zero $\star$-right ideal of $\mathcal{R}$. If U is commutative then $\mathcal{R}$ is commutative.

Proof. Since U, is commutative, by the Lemma 2.3, we have $U = Z(U) \subseteq Z(\mathcal{R})$. If for any $x, y \in \mathcal{R}$, $a \in U$ we have $ax \in U$ and hence $ax \in Z(\mathcal{R})$ and hence $(ax)y = y(ax) = ayx$. This further yields $U(xy - yx) = \{0\}$. Since U is a non-zero $\star$-right ideal of $\mathcal{R}$, we have $U\mathcal{R}(xy - yx) = \{0\} = U^\star\mathcal{R}(xy - yx)$. Also, since $U \neq \{0\}$ right ideal, $\star$-primeness of $\mathcal{R}$ gives $xy - yx = 0$, for all $x, y \in \mathcal{R}$. Hence $\mathcal{R}$ is commutative. □

Lemma 2.5. Let $\mathcal{R}$ be a $\star$-prime ring and U a non-zero $\star$-right ideal of $\mathcal{R}$. Suppose that $a \in \mathcal{R}$ centralizes U. Then $a \in Z(\mathcal{R})$.

Proof. Since a centralizes U, for all $u \in U$ and $x \in \mathcal{R}$, $aux = uxa$. But $au = ua$, therefore $uax = uxa$, i.e., $u[a,x] = 0$. On replacing u by uy for any $y \in \mathcal{R}$, we get $u\mathcal{R}[a,x] = \{0\}$ for all $u \in U$, $x \in \mathcal{R}$. Also, since U is $\star$-right ideal, we get $u^\star\mathcal{R}[a,x] = \{0\}$. Again since $U \neq \{0\}$, $\star$-primeness of $\mathcal{R}$ yields that $[a,x] = 0$ for all $x \in \mathcal{R}$. Therefore, $a \in Z(\mathcal{R})$. □

Lemma 2.6. Let $\mathcal{R}$ be a $\star$-prime ring and U a $\star$-right ideal of $\mathcal{R}$. Suppose d is a (σ,τ)-$\star$-derivation of $\mathcal{R}$ satisfying $d(U) = \{0\}$, then $d = 0$.

Proof. For all $u \in U$ and $x \in \mathcal{R}$, $0 = d(ux) = d(u)\sigma(x^\star) + \tau(u)d(x) = \tau(u)d(x)$. On replacing x by xy for any $y \in \mathcal{R}$, we get $\tau(u)d(x)\sigma(y^\star) + \tau(u)\tau(x)d(y) = 0$, or, $\tau(u)\tau(x)d(y) = 0$, i.e., $\tau(u)\mathcal{R}d(y) = \{0\}$ for all $u \in U$ and $y \in \mathcal{R}$. Also since U is a $\star$-right ideal, we get $\tau(u)^\star\mathcal{R}d(y) = \{0\}$. Also, $\star$-primeness of $\mathcal{R}$ yields that $\tau(u) = 0$ for all $u \in U$ or $d = 0$. Since $U \neq \{0\}$, we get $d = 0$. □

Lemma 2.7. Let $\mathcal{R}$ be a $\star$-prime ring, U a non-zero $\star$-ideal of $\mathcal{R}$ and $a \in \mathcal{R}$. Suppose d is a (σ,τ)-$\star$-derivation of $\mathcal{R}$ satisfying $ad(U) = \{0\}$ (or, $d(U)a = \{0\}$), then $a = 0$ or $d = 0$.

Proof. For $u \in U, x \in \mathcal{R}, 0 = ad(ux) = ad(u)\sigma(x^\star) + a\tau(u)d(x)$. By assumption, we have $a\tau(u)d(x) = 0$, for all $x \in \mathcal{R}$. On replacing u by uy for any $y \in \mathcal{R}$, we obtain $a\tau(u)\mathcal{R}d(x) = \{0\}$ for all $u \in U$, $x \in \mathcal{R}$. Also, $a\tau(u)\mathcal{R}d(x)^\star = \{0\}$. Since $\mathcal{R}$ is $\star$-prime, we find that either $a\tau(u) = 0$ or $d(x) = 0$. If $a\tau(u) = 0$ for all $u \in U$, then or $\tau^{-1}(a)U = \{0\}$. Now since U is $\star$-ideal, we can write $\tau^{-1}(a)U^\star = \{0\}$. This implies that $\tau^{-1}(a)\mathcal{R}U = \{0\} = \tau^{-1}(a)\mathcal{R}U^\star$. By the $\star$-primeness of $\mathcal{R}$, we obtain $\tau^{-1}(a) = 0$, since $U \neq \{0\}$. In conclusion, we get either $a = 0$ or $d = 0$. Similarly, $d(U)a = \{0\}$ implies $a = 0$ or $d = 0$. □

Lemma 2.8. Let d be a non-zero (σ,τ)-$\star$-derivation of $\star$-prime ring $\mathcal{R}$ and U a $\star$-right ideal of $\mathcal{R}$. If $d(U) \subseteq Z(\mathcal{R})$, then $\mathcal{R}$ is commutative.

Proof. Since $d(U) \subseteq Z(\mathcal{R})$, we have $[d(U),\mathcal{R}] = \{0\}$. For $u, v \in U$ and $x \in \mathcal{R}$,

$$[x, d(uv)] = [x, d(u)\sigma(v^\star) + \tau(u)d(v)] = d(u)[x,\sigma(v^\star)] + d(v)[x,\tau(u)] = 0. \quad (1)$$

Replacing x by $x\sigma(v^\star),\ v \in U$ in (1), we have

$$\begin{aligned} 0 &= d(u)[x\sigma(v^\star),\sigma(v^\star)] + d(v)[x\sigma(v^\star),\tau(u)] \\ &= d(u)[x,\sigma(v^\star)]\sigma(v^\star) + d(v)(x[\sigma(v^\star),\tau(u)] + [x,\tau(u)]\sigma(v^\star)). \end{aligned}$$

By using (1), we get

$$d(v)\mathcal{R}[\sigma(v^\star),\tau(u)] = \{0\},\ \text{for all } u, v \in U. \quad (2)$$

Let $v \in U \cap Sa_\star(\mathcal{R})$. From (2), it follows that

$$d(v)^\star\mathcal{R}[\sigma(v^\star),\tau(u)] = \{0\},\ \text{for all } u \in U. \quad (3)$$

By (2) and (3), the $\star$-primeness of $\mathcal{R}$ yields that $d(v) = 0$ or $[\sigma(v^\star), \tau(u)] = 0$ for all $u \in U$. Let $w \in U$, since $w - w^\star \in U \cap Sa_\star(\mathcal{R})$, then

$$d(w - w^\star) = 0 \text{ or } [\sigma(w - w^\star)^\star, \tau(u)] = 0.$$

Assume that $d(w - w^\star) = 0$. Then $d(w) = d(w^\star)$. Replacing v by $w^\star$ in (2) and since U is $\star$-right ideal, we get $d(w^\star)\mathcal{R}[\sigma(w^\star)^\star, \tau(u)] = \{0\}$ for all $u \in U$. Consequently,

$$d(w)\mathcal{R}[\sigma(w^\star), \tau(u)]^\star = \{0\}, \text{ for all } u, w \in U. \tag{4}$$

Also by (2), we get $d(w)\mathcal{R}[\sigma(w^\star), \tau(u)] = \{0\}$, on using $\star$-primeness of $\mathcal{R}$ together with (4), we find that for each $w \in U$ either $d(w) = 0$ or $[\sigma(w)^\star, \tau(u)] = 0$, for all $u \in U$. Now suppose the remaining case that $[\sigma(v)^\star, \tau(u)] = 0$, for all $u \in U$. Then we have $[\sigma(w - w^\star)^\star, \tau(u)] = 0 = [\sigma(w - w^\star), \tau(u)]$, or $[\sigma(w), \tau(u)] = [\sigma(w^\star), \tau(u)]$. Replacing v by $w^\star$ in (2), we get $d(w^\star)\mathcal{R}[\sigma(w^\star)^\star, \tau(u)] = \{0\}$ for all $u \in U$. Consequently, $d(w^\star)\mathcal{R}[\sigma(w), \tau(u)] = \{0\}$. This yields that

$$\text{or, } \ d(w^\star)\mathcal{R}[\sigma(w)^\star, \tau(u)] = \{0\}, \text{ for all } u, w \in U. \tag{5}$$

Since $d(w)\mathcal{R}[\sigma(w^\star), \tau(u)] = \{0\}$, by (2), the $\star$-primeness of $\mathcal{R}$ together with (5) assure that for each $w \in U$ either $d(w) = 0$ or $[\sigma(w^\star), \tau(u)] = 0$, for all $u \in U$. In conclusion, for each fixed $w \in U$, we have

$$\text{either } \ d(w) = 0 \text{ or } [\sigma(w^\star), \tau(u)] = 0 \text{ for all } u \in U.$$

Now, define

$$K = \{w \in U \mid d(w) = 0\} \text{ and } L = \{w \in U \mid [\sigma(w^\star), \tau(u)] = 0 \text{ for all } u \in U\}.$$

Clearly both K and L are additive subgroups of U whose union is U. But a group cannot be a set theoretic union of two of it's proper subgroups and hence either $K = U$ or $L = U$. If $K = U$, then $d(U) = \{0\}$ and hence by Lemma 2.6, $d = 0$, a contradiction, therefore now assume that $L = U$, i.e.,

$$[\sigma(w^\star), \tau(u)] = 0 \text{ for all } u, w \in U. \tag{6}$$

Replacing $w^\star$ by $w'\sigma^{-1}(\tau(v))$, $u \in U$, in (6) and using (6), we get $\sigma(w')\tau([v, u]) = 0$, for all $u, v, w' \in U$. On replacing w' by $w'x$ for any $x \in \mathcal{R}$, we get $\sigma(w')\mathcal{R}\tau([v, u]) = \{0\}$, for all $u, v, w' \in U$. Also, since U is $\star$-right ideal, we get $\sigma(w')^\star\mathcal{R}\tau([v, u]) = \{0\}$, for all $u, v, w' \in U$. Since $\mathcal{R}$ is $\star$-prime, we find that $\sigma(w') = 0$ or $\tau[v, u] = 0$ for all $u, v, w' \in U$. Since $U \neq \{0\}$, we have U is commutative. In view of Corollary 2.4, we obtain the commutativity of $\mathcal{R}$. □

We are now well equipped to prove our main theorem:

Proof of Theorem 2.2. First we will show that for any $a \in Sa_\star(\mathcal{R})$ such that $[d(U), a]_{\sigma,\tau} = \{0\}$, then $a \in Z(\mathcal{R})$. For any $v \in U$, using the hypothesis, we have

$$\begin{aligned} 0 &= [d(uv^\star), a]_{\sigma,\tau} \\ &= [d(u)\sigma(v) + \tau(u)d(v^\star), a]_{\sigma,\tau} \\ &= d(u)\sigma(v)\sigma(a) + \tau(u)d(v^\star)\sigma(a) - \tau(a)d(u)\sigma(v) - \tau(a)\tau(u)d(v^\star). \end{aligned}$$

In view of the hypothesis the above relation yields

$$d(u)\sigma([v,a]) + \tau([u,a])d(v^\star) = 0 \text{ for all } u, v \in U. \tag{7}$$

Replace u by au in (7) and use (7) to get

$$\begin{aligned} 0 &= d(au)\sigma([v,a]) + \tau([au,a])(d(v^\star) \\ &= \{d(a)\sigma(u^\star) + \tau(a)d(u)\}\sigma([v,a]) + \tau(a)\tau([u,a])d(v^\star). \end{aligned}$$

We have $d(a)\sigma(u^\star)\sigma([v,a]) = 0$, for all $u, v \in U$. Replace $u^\star$ by xw for any $x \in \mathcal{R},\ w \in U$ we find that $d(a)\mathcal{R}\sigma(w)\sigma([v,a]) = \{0\}$, for all $w, v \in U$. Since $a \in Sa_\star(\mathcal{R})$, the above expression can be rewritten as $d(a)^\star\mathcal{R}\sigma(w)\sigma([v,a]) = \{0\}$, for all $u, v \in U$. On using $\star$-primeness of $\mathcal{R}$, we obtain that for all $u, v \in U$

$$\sigma(w)\sigma([v,a]) = 0 \text{ or } d(a) = 0. \tag{8}$$

Let us suppose that $d(a) = 0$. Then for all $u \in U$,

$$\begin{aligned} d([u,a^\star]) &= d(ua^\star - a^\star u) \\ &= d(u)\sigma(a) + \tau(u)d(a^\star) - d(a^\star)\sigma(u^\star) - \tau(a^\star)d(u) \\ &= d(u)\sigma(a) - \tau(a^\star)d(u) - \tau(a)d(u) + \tau(a)d(u) \\ &= [d(u), a]_{\sigma,\tau} + \tau(a - a^\star)d(u) \\ &= \tau(a - a^\star)d(u). \end{aligned}$$

Hence the above yields that

$$d([u,a^\star]) - \tau(a - a^\star)d(u) = 0. \tag{9}$$

On replacing u by uv, $v \in U$, in (9) and on using the same, we get

$$\tau([u,a^\star])d(v) + d(u)\sigma([v,a^\star])^\star + \tau(u)d([v,a^\star]) - \tau(a - a^\star)\tau(u)d(v) = 0.$$

By using (9), for all $u, v, w \in U$ we have

$$\begin{aligned} 0 &= \tau([u,a^\star])d(v) + d(u)\sigma([v,a^\star])^\star \\ &\quad + \tau(u)\tau(a - a^\star)d(v) - \tau(a - a^\star)\tau(u)d(v) \\ &= \tau([u,a^\star])d(v) + d(u)\sigma([v,a^\star])^\star + \tau([u, a - a^\star])d(v) \\ &= \tau([u,a])d(v) + d(u)\sigma([v,a^\star])^\star. \end{aligned}$$

Again by using (7), we have

$$\begin{aligned} 0 &= -d(u)\sigma([v^\star, a]) + d(u)\sigma([v, a^\star])^\star \\ &= 2d(u)\sigma([a, v^\star]). \end{aligned}$$

Since *char* $\mathcal{R} \neq 2$, we get $d(u)\sigma([a, v^\star]) = 0$ for all $u, v \in U$. Replacing $v^\star$ by w in the above relation, we get $d(u)\sigma([a, w]) = 0$ for all $u, w \in U$. Substituting w by ww' for any $w' \in U$, reduces the above relation to $d(u)U\sigma([a, w']) = \{0\}$ for all $u, v, w \in U$, or $\sigma^{-1}(d(u))U[a, w'] = \{0\}$ for all $u, v, w \in U$. Therefore,

$$\sigma^{-1}(d(u))\mathcal{R}U[a, w'] = \{0\} \text{ for all } u, v, w \in U.$$

Since U is a $\star$-ideal, using $\star$-primeness of $\mathcal{R}$, we get either $\sigma^{-1}(d(u)) = 0$ for all $u \in U$ or $U[a, w'] = \{0\}$ for all $w' \in U$. Since $d(U) \neq \{0\}$, we have $U[a, w'] = \{0\} = U\mathcal{R}[a, w']$. Since U is a nonzero $\star$-ideal, using $\star$-primeness of $\mathcal{R}$, we get $[a, w'] = 0$, for all $w' \in U$. This reduces to $[U, a] = \{0\}$. In view of Lemma 2.5, we find that $a \in Z(\mathcal{R})$. In view of (8) consider the remaining part $\sigma(w)\sigma([v, a]) = 0$ for all $w, v \in U$, i.e., $w[v, a] = 0$ for all $w, v \in U$. On replacing w by wx for any $x \in \mathcal{R}$, the above equation reduces to $w\mathcal{R}[v, a] = \{0\}$, for all $w, v \in U$. Also, U being a $\star$-ideal, we get $w^\star\mathcal{R}[v, a] = \{0\}$. Using the $\star$-primeness of $\mathcal{R}$ we find that either $[v, a] = \{0\}$ or $U = \{0\}$. Since $U = \{0\}$ is not possible, it reduces to $[U, a] = \{0\}$. Hence again in view of Lemma 2.5, we find that $a \in Z(\mathcal{R})$, and by our hypothesis we obtain that $d(U) \subseteq Z(\mathcal{R})$. So by Lemma 2.8, $\mathcal{R}$ is commutative. □

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