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ON AN UPPER NIL RADICAL FOR NEAR-RING MODULES

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Abstract

For a near-ring R we introduce the notion of an $s-$prime $R-$module and an $s-$system. We show that an $R-$ideal P is an $s-$prime $R-$ideal if and only if $R\backslash P$ is an $s-$system. For an $R-$ideal N of the near-ring module M we define $\mathcal{S}(N) =: \{m \in M$: every $s-$sytem containing m meets $N\}$ and prove that it coincides with the intersection of all the $s-$prime $R-$ideals of M containing N. $\mathcal{S}(0)$ is an upper nil radical of the near-ring module. Furthermore, we define a $\mathcal{T}-$special class of near-ring modules and then show that the class of $s-$prime modules forms a $\mathcal{T}-$special class. $\mathcal{T}-$special classes of $s-$prime near-ring modules are then used to describe the 2-s-prime radical of a near-ring.

1 Introduction

In 1961 Andrunakievič [1] introduced the notion of a prime module ${}_RM$ over an associative ring R and then used the notion of a prime module to characterize the prime radical of the ring R. In 1964 Andrunakievič and Rjabuhin [3] used the notion of a prime module to define special classes of modules and then used the notion of a special class of $R-$modules to characterize special classes of rings and special radicals. In 1978 Dauns [8] was the first to do a detailed study of prime modules. The notion of a prime near-ring module was briefly introduced by Beidleman in 1967 in [5]. Later equiprime, strongly prime and different

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types of prime near-ring modules were introduced by Booth and Groenewald [9] and also by Groenewald, Juglal and Lee [13].

2 Prime and $s-$prime near-rings

For the near-ring R and a subset K of R, $\langle K \mid_R$, $\mid K\rangle_R$, $\langle K\rangle_R$, $\langle K]_R$ and $[K\rangle_R$ denote the left ideal, right ideal, two-sided ideal, left $R-$subgroup and right $R-$subgroup generated by K in R respectively. If it is clear in which near-ring we are working, the subscript R will be omitted. Also $K \lhd_l R$, $K \lhd_r R$, $K \lhd R$ and $K <_R R$ symbolize that K is a left ideal, right ideal, two-sided ideal or a left $R-$subgroup of R.

Definition 2.1. Let R be a near-ring (not necessarily zero-symmetric) and P an ideal of R.

1. P is a **0-prime** ideal if for every A, $B \lhd R$, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ (this is the same as the usual definition for a prime ideal in a ring).

2. P is a $2-$**prime** ideal if for every A and B left R-subgroups of R, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

R is called an i**-prime near ring** ($i = 0, 2$) if the zero ideal is an i-prime ideal.

Let $\mathcal{M}$ be any class of near-rings and ρ a mapping which assigns to each near-ring R an ideal $\rho(R)$ of R. If $\rho(R) = \cap\{I \lhd R : R/I \in \mathcal{M}\}$, then ρ is a Hoehnke radical (H-radical), called the H-radical associated with the class $\mathcal{M}$. In [19] van der Walt defined the notion of an s-prime near-ring and showed that the Hoehnke radical determined by the class of all s-prime near-rings is the same as the upper nil radical. Hence if R is a near-ring then $\mathcal{N}(R)$ i.e., the sum of all nil ideals of R is equal to $s(R)$ the intersection of all the s-prime ideals of R (all ideals I such that R/I is an s-prime near-ring). In [16] Kaarli observed that the nil radical $\mathcal{N}(R)$ of the near-ring R is equal to the intersection of all the 0-prime ideals P of R such that R/P has no nonzero nil ideals. He mentioned that the proof of this result is essentially that given for rings by Divinsky, see [7, page 147]. In [6] Birkenmeier et al called an ideal I of the near-ring R nilprime if I is a 0-prime ideal and $\mathcal{N}(R/I) = 0$ i.e., R/I has no nonzero nil ideals. They then gave a selfcontained proof within near-ring theory of the above result mentioned by Kaarli. In [6] it was proved that every s-prime near-ring is a nilprime near-ring.

Definition 2.2. [11, Definition 3.1] A near-ring is i-nilprime if R is i-prime and R contains no nonzero nil ideals and $i \in \{0, 2\}$.

From [11, Theorem 2.7 and Proposition 2.12] we have that the upper nil radical of the near-ring R is equal to the intersection of all the $0-$nilprime ideals.

If R is an associative ring, this coincides with the notion prime nil-semisimple rings and the upper radical determined by this class of rings coincides with the nilradical $\mathcal{N}(R)$. In the case of near-rings this give rise to nonequivalent nil-radicals.

Example 2.3. [11, Example 3.2]Let G be a finite group and let $0 \neq H$ be a proper subgroup of G. Let $R = \{a \in M_0(G) : a(H) \subseteq H\}$. Then R is a zero-symmetric near-ring. R is 0-nilprime but not 2-nilprime.

Definition 2.4. [11, Definition 4.1] A subset T of the near-ring R is called a complete system if $a^n \in T$ for every $a \in T$ and every $n \in \mathbb{N}$.

If $a, b \in R$ we will use the following notation:

$$[a]^i\,[b]^i = \begin{cases} \langle a\rangle\langle b\rangle & \text{if } i=0 \\ < a]_R < b]_R & \text{if } i=2 \end{cases}$$

Note that an ideal Q of R is i-prime, $i \in \{0,2\}$, if for $a, b \in R$, $[a]^i\,[b]^i \subseteq Q$ implies $a \in Q$ or $b \in Q$.

Definition 2.5. [11, Definition 4.2] A subset $Z \subseteq R$ is called an $i-s$-system, $i \in \{0,2\}$, if Z contains a complete system U such that for every $t_1, t_2 \in Z$, it follows that $[t_1]^i\,[t_2]^i \cap U \neq \varnothing$.

Definition 2.6. [11, Definition 4.3] An ideal Q is called $i-s$-prime, $i \in \{0,2\}$, if for $a, b \in R$ and for all $x \in [a]^i\,[b]^i$, $x^m \in Q$ for some m implies $a \in Q$ or $b \in Q$.

Proposition 2.1. *[11, Proposition 4.4] An ideal Q of R is $i-s$-prime, $i \in \{0,2\}$, if and only if $\mathcal{C}_R\,(Q)$ is an $i-s$-system where $\mathcal{C}_R\,(Q)$ is the complement of Q in R.*

Proposition 2.2. *[11, Proposition 4.5] An ideal Q of R is $i-s$-prime if and only it is $i-$nilprime, $i \in \{0,2\}$.*

By a similar argument as in [11, Lemma 4.5] we get the following:

Lemma 2.7. *Let V be a non-void $2-s$ system in R and I an ideal of R such that $V \cap I = \varnothing$. Then I is contained in a $2-s$-prime ideal $P \neq R$ with $V \cap P = \varnothing$.*

Definition 2.8. The $i-s$-radical $i \in \{0,2\}$ of R, denoted by $s_i(R)$, consists of all those elements $r \in R$ such that every $i-s$-system which contains r also contains 0.

By a similar argument as in [11, Theorem 2.7] we get the following:

Theorem 2.9. *The $2-s$-radical $s_2(R)$ of the near ring R is equal to the intersection of all the $2-s$-prime ideals of R.*

3 Prime near-ring modules

Let R be a near-ring, M any left $R-$module and P a subset of M. If P is an $R-$ideal ($R-$submodule) of M we denote it by $P \lhd_R M$ ($P \leq_R M$). If it is clear in which near-ring we are working, the subscript R will be omitted.

Definition 3.1. Let P be an $R-$ideal of an $R-$module M such that $RM \nsubseteq P$. Then P is a prime $R-$ideal if for any ideal A of R and any submodule N of M, $AN \subseteq P$ implies $AM \subseteq P$ or $N \subseteq P$. M is said to be a prime $R-$module if $RM \neq 0$ and 0 is a prime $R-$ideal.

Example 3.2. Let R be a near-ring with identity and M an $R-$module with no proper nonzero $R-$submodules, then M is a prime module.

Example 3.3. Let F be a field and $R = F \times F$. $M = 0 \times F$ has no proper $R-$submodules and hence M is a prime module.

Example 3.4. Every $R-$module of M of type 2 is prime: Let $A \lhd R$ and $N \leq M$ such that $AN = 0$. If $N = 0$ then we are done. If $N \neq 0$, then $N = M$ and we have $AM = 0$.

Proposition 3.1. *Let R be a zero symmetric near-ring. If M is an $R-$module and P is an $R-$ideal of M, then the following are equivalent:*

1. *P is a prime $R-$ideal.*
2. *For all $a \in R$ and $m \in M$, $\langle a\rangle m \subseteq P$ implies $\langle a\rangle M \subseteq P$ or $m \in P$ where $\langle a\rangle$ is the ideal of R generated by a.*
3. *For all $a \in R$ and $m \in M$, $a\langle m\rangle_R \subseteq P$ implies $aM \subseteq P$ or $m \in P$ where $\langle m\rangle_R$ is the submodule of M generated by m.*
4. *$\mathcal{P} = (P : M)$ is a $2-$prime ideal of R and $(P : M) = (P : \langle m\rangle_R)$ for all $m \notin P$.*
5. *$\{(P : \langle m\rangle_R) : m \in M\backslash P\}$ is a singleton.*

Proof . 1. $\Rightarrow$ 2. Let $a \in R$ and $m \in M$ such that $\langle a\rangle m \subseteq P$. Hence $\langle a\rangle Rm \subseteq P$. Since P is a prime $R-$ideal it follows that $\langle a\rangle M \subseteq P$ or $Rm \subseteq P$ i.e. $\langle a\rangle M \subseteq P$ or $m \in P$.

2. $\Rightarrow$ 3. Let $a \in R$ and $m \in M$ such that $a\langle m\rangle_R \subseteq P$. Hence $a \in (P : \langle m\rangle_R)$ and since $(P : \langle m\rangle_R)$ is an ideal of R we have $\langle a\rangle \subseteq (P : \langle m\rangle_R)$. Now $\langle a\rangle m \subseteq P$ and it follows from 2. that $\langle a\rangle M \subseteq P$ or $m \in P$. Hence $aM \subseteq P$ or $m \in P$.

3. $\Rightarrow$ 4. Let K and L be $R-$subgroups such that $KL \subseteq (P : M)$. If $L \subseteq (P : M)$, then $(P : M)$ is a $2-$prime ideal of R. Suppose $LM \nsubseteq P$, say $lm \notin P$ for some $l \in L$ and $m \in M$ i.e. $\langle lm\rangle_R \nsubseteq P$. Since $K\langle lm\rangle_R \subseteq P$, it follows

from 3. that $KM \subseteq P$ and we have we have $K \subseteq (P : M)$. Hence, $(P : M)$ is a $2-$prime ideal of R. To show that $(P : M) = (P : \langle m \rangle_R)$, we only have to show $(P : \langle m \rangle_R) \subseteq (P : M)$. Let $y \in (P : \langle m \rangle_R)$. Since $y\langle m \rangle_R \subseteq P$ and $m \notin P$, it follows from 3. that $yM \subseteq P$. Hence $y \in (P : M)$ and we have $(P : M) = (P : \langle m \rangle_R)$.

4. $\Rightarrow$ 5. This is clear.

5. $\Rightarrow$ 1. Let A be an ideal of R and N a submodule of M such that $AM \nsubseteq P$ and $N \nsubseteq P$. We show that $AN \nsubseteq P$. There exists $m \in M \backslash P$ such that $A\langle m \rangle_R \nsubseteq P$, that is $A \nsubseteq (P : \langle m \rangle_R)$. Now, if $n \in N \backslash P$ is arbitrary, we have from 5. that $A \nsubseteq (P : \langle n \rangle_R) = (P : \langle m \rangle_R)$, that is $A\langle n \rangle_R \nsubseteq P$. Hence $AN \nsubseteq P$ and therefore P is a prime $R-$ideal. □

Proposition 3.2. *Let R be a zero symmetnc near-ring and let $\mathcal{P} \triangleleft R$. There is a prime $R-$module M with $(0 : M)_R = \mathcal{P}$ if and only if $\mathcal{P}$ is a $2-$prime ideal of R.*

Proof. Let $\mathcal{P}$ be a $2-$prime ideal of R and let $M = R/\mathcal{P}$. M is an $R-$module with the natural operation. Clearly, $\mathcal{P} \subseteq (0 : M)_R$. Let $a \in (0 : M)_R$. Hence $a(r + \mathcal{P}) = \mathcal{P}$ for each $r \in R$. Then, $aR \subseteq \mathcal{P}$ and since $\mathcal{P}$ is a $2-$prime ideal, we have $a \in \mathcal{P}$ and $\mathcal{P} = (0 : M)_R$. $RM \neq 0$ for if $RM = 0$, then $R^2 \subseteq \mathcal{P}$ and since $\mathcal{P}$ is $2-$prime ideal, we get $R \subseteq P$ which is not possible. M is a prime $R-$module. Let $A \triangleleft R$ and $J \leq_R M$ such that $AJ = 0$. Now $J = L/\mathcal{P}$ for some left $R-$subgroup L of R containing $\mathcal{P}$. Hence, $AL \subseteq \mathcal{P}$ and since $\mathcal{P}$ is a $2-$prime ideal we have $A \subseteq \mathcal{P}$ or $L \subseteq \mathcal{P}$. Thus, $AM = 0$ or $J = 0$ and we are through. Let M be a prime $R-$module and A, $B \leq R$ such that $AB \subseteq (0 : M)_R$. Suppose $B \nsubseteq (0 : M)_R$ i.e. $BM \neq 0$. Then, there exists $m \in M$ such that $Bm \neq 0$. Now $ABm = 0$ and also $\langle A \rangle Bm = 0$. Since M is a prime $R-$module, $AM \subseteq \langle A \rangle M = 0$. Hence $(0 : M)_R$ is a $2-$prime ideal. □

Remark 3.5. The above proposition gives a method to construct prime $R-$modules. For any $2-$prime ideal $\mathcal{P}$ of R, $M = R/\mathcal{P}$ is a prime $R-$module.

4 $s-$prime $R-$modules

Definition 4.1. A proper $R-$ideal P of M with $RM \nsubseteq P$ is called an $s-$prime $R-$ideal if the following is satisfied: If for every $A \triangleleft R$ and every $N \leq M$ if $x \in A$ and $x^n N \subseteq P$ for some $n \in \mathbb{N}$, then $N \subseteq P$ or $AM \subseteq P$. M is said to be an $s-$prime $R-$ module if $RM \neq 0$ and 0 is an $s-$prime $R-$ideal.

Example 4.2. Let R be the near-ring on $\mathbb{Z}_3 = \{0, 1, 2\}$ with multiplication defined by: $a \cdot b = \begin{cases} a & \text{if} \quad b = 2 \\ 0 & \text{if} \quad b \neq 2 \end{cases}$. The only $R-$subgroups of R are 0 and R. We also have $R^2 \neq 0$. Hence R is $2-$prime. Furthermore we have $2^n = 2$ for every $n \in \mathbb{N}$. Thus $M =_R R$ is an $s-$prime module.

Example 4.3. Every type 2 $R-$module M is an $s-$prime $R-$module: Suppose $N \neq 0$ and $AM \neq 0$ for some $A \lhd R$ and $N \leq {}_RM$. Since M is monogenic, there exists $m \in M$ such that $M = Rm$. Now, since $AM \neq 0$, we have $0 \neq ARm \subseteq Am$. Since Am is a submodule of M and M is of type 2, $Am = M$ and there exists $a \in A$ such that $am = m$. Hence $a^k m = m$ for every $k \in \mathbb{N}$. Consequently $0 \neq m \in a^k M = a^k N$ for every $k \in \mathbb{N}$ and therefore M is $s-$prime

Example 4.4. A prime module which is not an $s-$prime module. We use the construction and computation in [15, Example 1.2 and Proposition 1.3]. Let S be a domain, n be a positive integer and R_n be the 2^n by 2^n upper triangular matrix ring over S. Define a map $\delta : R_n \to R_{n+1}$ by $A \to \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$. Then R_n is considered as a subring of R_{n+1} via δ. $D_n = \{R_n, \delta_{nm}\}$, with $\delta_{nm} = \delta^{m-n}$ whenever $n \leq m$, is a direct system over $I = \{1, 2, 3, \cdots, \}$. Let $R = \underrightarrow{lim} R_n$ be the direct limit of D_n. From [15], R is a prime near-ring with $\mathcal{N}(R) \neq 0$. Hence a prime near-ring which is not an $s-$prime near-ring. If we let $M = {}_RR$, then M is a prime $R-$module which is not an $s-$prime $R-$module.

Proposition 4.1. *An $R-$ideal $P \lhd_R M$ is $s-$prime if and only if:*

1. *P is a prime $R-$ideal.*
2. *for every $A \lhd R$ such that $A \not\subseteq (P : M)$ there exists $a \in A \setminus (P : M)$ such that $a^n M \not\subseteq P$ for all $n \in \mathbb{N}$.*

Proof. ($\Rightarrow$).

1. Let $A \lhd R$, $N \leq M$ such that $AN \subseteq P$. Hence for all $x \in A$, we have $xN \subseteq P$. Since P is $s-$prime we have $N \subseteq P$ or $AM \subseteq P$. Hence P is a prime $R-$ideal.

2. Let $B \lhd R$ such that $B \not\subseteq (P : M)$, i.e. $BM \not\subseteq P$. Let $m \in M$ such that $Bm \not\subseteq P$. Now, since P is an $s-$prime R-ideal, there exists $b \in B$ such that $b^n Bm \not\subseteq P$ for all $n \in \mathbb{N}$. Hence $b^n M \not\subseteq P$ for every $n \in \mathbb{N}$.

($\Leftarrow$). Let $A \lhd R$ and $N \leq M$. Suppose $N \not\subseteq P$ and $AM \not\subseteq P$. Let $n \in N$ such that $n \notin P$. Now, since P is prime, we have $(P : M) = (P : \langle n \rangle)$ is a 2-prime ideal. From 2., there exists $a \in A$ such that $a^k M \not\subseteq P$ for every $k \in N$. Hence $a^k \notin (P : M) = (P : \langle n \rangle)$ for very $k \in \mathbb{N}$. Thus $a^k \langle n \rangle \not\subseteq P$ and $a^k N \not\subseteq P$ for every $k \in \mathbb{N}$, so P is an $s-$prime R -ideal. □

Corollary 4.5. *An $R-$ideal $P \lhd_R M$ is $s-$prime if and only if*

(a) P is a prime $R-$ideal and

(b) $R/(P : M)$ contains no nonzero nil ideals i.e. $\mathcal{N}(R/(P : M)) = 0$ where $\mathcal{N}(R)$ is the upper nil radical of the near-ring R.

Proposition 4.2. *If R is a commutative or an Artinian near-ring, then an $R-$ideal $P \lhd_R M$ is $s-$prime if and only if P is prime.*

Proof. Suppose $P \lhd_R M$ is a prime $R-$ideal of M. From Proposition 3.1 we have $(P : M)$ a 2-prime ideal and therefor also a 0-prime ideal of R. If$\wp(R)$ denotes the $0-$prime radical of the near-ring R then $0 = \wp(R/(P : M)) = \mathcal{N}(R/(P : M))$ and from Corollary 4.5 we have P is $s-$prime. The converse is clear. □

Proposition 4.3. *Let M be an $R-$module. For a proper $R-$ideal $P \lhd_R M$, the following statements are equivalent:*

1. *P is an $s-$prime $R-$ideal of M;*
2. *for every $a \in R$ and for every $m \in M$, if $a^n\langle m\rangle \subseteq P$ for some $n \in \mathbb{N}$, then $m \in P$ or $\langle a\rangle M \subseteq P$;*
3. *P is a prime $R-$ideal and for every $a \in R$ such that $aM \nsubseteq P$ we have that that $a^n M \nsubseteq P$ for every $n \in \mathbb{N}$;*
4. *$\mathcal{P} = (P : M)$ is a $2-s-$prime ideal of R and $(P : \langle m\rangle) = \mathcal{P}$ for every $m \in M \setminus P$.*

Proof. 1. $\Rightarrow$ 2. Suppose P is an $s-$prime $R-$ ideal of M and for all $a \in R$ and for every $m \in M$ we have $a^n\langle m\rangle \subseteq P$ for some $n \in \mathbb{N}$. Hence for every $x \in \langle a\rangle \lhd R$ and $m \in \langle m\rangle_R \leq M$ we have $x^t\langle m\rangle_R \subseteq P$ for some $t \in \mathbb{N}$. Since P is an $s-$prime $R-$ ideal we have $\langle a\rangle M \subseteq P$ or $\langle m\rangle_R \subseteq P$. Thus $m \in P$ or $\langle a\rangle M \subseteq P$ and we are done.

2. $\Rightarrow$ 3. Let $a \in R$ and $m \in M$ such that $a\langle m\rangle_R \subseteq P$. From 2. we have $m \in P$ or $\langle a\rangle M \subseteq P$. Hence P is a prime $R-$ideal. If there exists $b \in R$ such that $bM \nsubseteq P$ then $\langle b\rangle M \nsubseteq P$. It now follows from 2 that if $m \in M\backslash P$ then $b^n\langle m\rangle \nsubseteq P$ for every $n \in \mathbb{N}$. Hence $b^n M \nsubseteq P$ for every $n \in \mathbb{N}$.

3. $\Rightarrow$ 4. Since P is a prime $R-$ideal, we have from Proposition 3.1 4. that $\mathcal{P} = (P : M)$ is a 2-prime ideal of R and $\mathcal{P} = (P : M) = (P : \langle m\rangle)$ for every $m \in M \setminus P$. We now show that $\mathcal{N}(R/(P : M)) = 0$. Let $A \lhd R$ such that $A \nsubseteq (P : M)$. Suppose $a \in A$ such that $aM \nsubseteq P$. From 3. $a^n M \nsubseteq P$ for every $n \in \mathbb{N}$ and we have $\mathcal{N}(R/(P : M)) = 0$. It now follows from Proposition 2.2 that $(P : M)$ is a $2-s-$prime ideal of R.

4. $\Rightarrow$ 1. Suppose $\mathcal{P} = (P : M)$ is a $2-s-$prime ideal of R and $(P : \langle m\rangle) = \mathcal{P}$ for every $m \in M \setminus P$. Because $(P : M)$ is a 2-prime ideal of R and $(P : \langle m\rangle) = (P : M)$ for every $m \in M \setminus P$ it follows from Proposition 3.1 5. that P is a prime $R-$ideal.of M. Furthermore, since $(P : M)$ is a $2-s-$prime ideal of R we have from Proposition 2.2 that $\mathcal{N}(R/(P : M)_R) = (0)$. It now follows from Corollary 4.5 that P is an $s-$prime $R-$ideal. □

Proposition 4.4. *Let R be a zero symmetric near-ring and $\mathcal{P} \lhd R$, $\mathcal{P} \neq R$. The following are equivalent:*

1. *$\mathcal{P}$ is a $2-s-$prime ideal of R;*

2. *There exists a s-prime $R-$module M such that $\mathcal{P}=(0:M)_R$.*

Proof. Let $\mathcal{P}$ be a $2-s-$prime ideal and let $M=R/\mathcal{P}$. M is an $R-$module with the usual operation. Since $\mathcal{P}$ is a $2-$prime ideal, M is a prime module. Now, since $\mathcal{P}$ is a $2-s-$prime ideal of R we have from Proposition 2.2 that $\mathcal{N}(R/\mathcal{P})=\mathcal{N}(R/(0:M)_R)=(0)$. It now follows from Corollary 4.5 that (0) is an $s-$prime $R-$ideal of M. Hence M is an $s-$prime module.

2. $\Rightarrow$ 1. Follows from Proposition 4.3. $\square$

5 $s-$systems and the $s-$prime radical of a Module

Definition 5.1. Let R be a near-ring and M an $R-$module. A nonempty set $S\subseteq M\backslash\{0\}$ is called an $s-$system if, for each ideal A of R and for all submodules K, L of M, if $(K+L)\cap S\neq\varnothing$ and $(K+AM)\cap S\neq\varnothing$, then there exists $x\in A$ such that $(K+x^nL)\cap S\neq\varnothing$ for every $n\in\mathbb{N}$.

Corollary 5.2. *Let M be an $R-$module. Then, the $R-$ideal P of M is $s-$prime if and only of $M\setminus P$ is an $s-$system.*

Proof. $(\Rightarrow)$. Suppose $S=M\setminus P$. Let $A\triangleleft R$ and K, L submodules of M such that $(K+L)\cap S\neq\varnothing$ and $(K+AM)\cap S\neq\varnothing$. Suppose that for every $x\in A$ there exist an $n\in\mathbb{N}$ such that $(K+x^nL)\cap S=\varnothing$. Hence for every $x\in A$ there exists $n\in\mathbb{N}$ such that $x^nL\subseteq P$. Since P is $s-$prime, we have $L\subseteq P$ or $AM\subseteq P$. It follows that $(K+L)\cap S=\varnothing$ or $(K+AM)\cap S=\varnothing$, a contradiction. Therefore, S is an $s-$system.

$(\Leftarrow)$. Suppose that for every $A\triangleleft R$, $L\leq M$ and $x\in A$ there exists $n\in\mathbb{N}$ such that $x^nL\subseteq P$. If $L\nsubseteq P$ and $AM\nsubseteq P$, then $L\cap S\neq\varnothing$ and $AM\cap S\neq\varnothing$. Hence, there exists $a\in A$ such that $a^nL\cap S\neq\varnothing$ for every $n\in\mathbb{N}$. Thus we get an $a\in A$ such that $a^nL\nsubseteq P$ for every $n\in\mathbb{N}$, a contradiction. Therefore, P is an $s-$prime $R-$ideal. $\square$

Proposition 5.1. *Let M be an $R-$module, P a proper $R-$ideal of M and let $S=:M\setminus P$. Then, the following statements are equivalent:*

1. *P is $s-$prime;*

2. *S is an $s-$system;*

3. *For every $A\triangleleft R$ and for all $L\leq M$, if $L\cap S\neq\varnothing$ and $AM\cap S\neq\varnothing$, then there exists $a\in A$ such that $a^nN\cap S\neq\varnothing$ for every $n\in\mathbb{N}$;*

4. *For every $a \in R$ and every $m \in M$, if $\langle m\rangle \cap S \neq \varnothing$ and $aM \cap S \neq \varnothing$, then $a^n\langle m\rangle \cap S \neq \varnothing$ for all $n \in \mathbb{N}$.*

Proof. 1. $\Leftrightarrow$ 2. follows from Corollary 5.2. 2. $\Rightarrow$ 3. $\Rightarrow$ 4. is clear.

4. $\Rightarrow$ 1. Suppose $a \in R$ and $m \in M$ such that $a^n\langle m\rangle \subseteq P$ for some $n \in \mathbb{N}$. Suppose $\langle m\rangle \nsubseteq P$ and $aM \nsubseteq P$. Now, $\langle m\rangle \cap S \neq \varnothing$ and $aM \cap S \neq \varnothing$ and from 4. $a^t\langle m\rangle \cap S \neq \varnothing$ for all $t \in \mathbb{N}$. Hence also $a^n\langle m\rangle \cap S \neq \varnothing$, i.e., $a^n\langle m\rangle \nsubseteq P$ a contradiction. It now follows from Proposition 4.3 that P is an $s-$prime $R-$ideal. □

Proposition 5.2. *Let M be an $R-$module $S \subseteq M$ an $s-$system and P an $R-$ideal of M maximal with respect to the property that $P \cap S = \varnothing$. Then P is an $s-$prime $R-$ideal.*

Proof. Suppose $A \triangleleft R$ and $L \leq M$ such that for every $a \in A$, $a^n L \subseteq P$ for some $n \in \mathbb{N}$. If $L \nsubseteq P$ and $AM \nsubseteq P$ then $(L+P) \cap S \neq \varnothing$ and $(AM+P) \cap S \neq \varnothing$. Since S is an $s-$system, there exists $b \in A$ such that $(b^k L + P) \cap S \neq \varnothing$ for every $k \in \mathbb{N}$. Since $b^t L \subseteq P$ for some $t \in \mathbb{N}$, we have $P \cap S \neq \varnothing$, a contradiction. Hence, P must be an $s-$prime $R-$ideal. □

Definition 5.3. Let R be a near-ring and M and $R-$module. For an $R-$ideal N of M, if there is an $s-$prime $R-$ideal containing N, then we define

$$\mathcal{S}(N) =: \{m \in M : \text{every } s-\text{system containing } m \text{ meets } N\}$$

We write $S(N) = M$ whenever there is no $s-$prime $R-$ideal of M containing N.

Theorem 5.4. *Let M be an $R-$module and N an $R-$ideal of M. Then, either $S(N) = M$ or $S(N)$ equals the intersection of all $s-$prime $R-$ideals of M containing N.*

Proof. Suppose $S(N) \neq M$. This means

$$\{P : P \text{ is an } s-\text{prime } R-\text{ideal of } M \text{ and } N \subseteq P\} \neq \varnothing$$

We first prove $S(N) \subseteq \cap\{P : P$ is an $s-$prime $R-$ideal of M and $N \subseteq P\}$. Let $t \in S(N)$ and P any $s-$prime $R-$ideal of M with $N \subseteq P$. Consider the $s-$system $M \setminus P$. This $s-$system cannot contain t, for otherwise it meets N and hence also P. Hence $t \in P$. Conversely, assume $t \notin S(N)$, then there exists an $s-$system S such that $t \in S$ and $S \cap N = \varnothing$. From Zorn's Lemma, there exists an $R-$ideal $P \supseteq N$ which is maximal with respect to $P \cap S = \varnothing$. From Proposition 5.2, P is an $s-$prime $R-$ideal of M and $t \notin P$, as desired. □

In what follows, let R be any zero symmetric near-ring (need not have an identity). If M is an $R-$module, define
$\mathcal{S}({}_RM) = \cap \{P \lhd_R M : P \text{ is an } s-\text{prime} R - \text{ideal of} M\}$.
For the near-ring R, consider the $R-$module ${}_RR$. We have the following:

Lemma 5.5. $\mathcal{S}({}_RR) \subseteq s_2(R)$.

Proof. Let $x \in \mathcal{S}({}_RR)$ and let $I \lhd R$ be a $2-s-$prime ideal of R. From Proposition 4.3, we have R/I is an $s-$prime $R-$module. Hence, $x \in I$ and we have $x \in s_2(R)$, i.e., $\mathcal{S}({}_RR) \subseteq s_2(R)$. □

Remark 5.6. In general this containment is strict: Since every ring is also a near-ring, this follows from [14, Example 5.1].

Lemma 5.7. *For any near-ring R and any $R-$module M we have*

$$s_2(R) \subseteq (\mathcal{S}({}_RM) : M)_R$$

Proof. We have $(\mathcal{S}({}_RM) : M)_R = ([\bigcap\{S \lhd_R M : S\ s-\text{prime}\}] : M) = \bigcap\{(S : M)\ S \lhd_R M : S\ s-\text{prime}\}$. Since $(S : M)_R$ is a $2-s-$prime ideal for each $s-$prime R-ideal S of M we get $s_2(R) \subseteq (\mathcal{S}({}_RM) : M)_R$. □

The containment is strict: let $R = \mathbb{Z}$ and $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}$ for some prime number p. $\mathcal{S}_R(M) = \mathbb{Z}_{p^\infty}$ and $s_2(R) = (0)$, i.e., $s_2(R)M = (0)$.

Proposition 5.3. *For any near-ring R, $s_2(R) = (\mathcal{S}({}_RR) : R)_R$.*

Proof. From Lemma 5.7 $s_2(R) \subseteq (\mathcal{S}_RR) : R)_R$. Since $\mathcal{S}({}_RR) \subseteq s_2(R)$ we have $s_2(R) \subseteq (\mathcal{S}({}_RR) : R) \subseteq (s_2(R) : R)$. Let $x \in (s_2(R) : R)$. Hence $xR \subseteq s_2(R) = \bigcap\{\mathcal{P} \lhd R : \mathcal{P}\ 2-s-\text{prime}\}$. Since $xR \subseteq \mathcal{P}$ for each $2-s-$prime ideal $\mathcal{P}$ and since $\mathcal{P}$ is a 2-prime ideal we have $x \in \mathcal{P}$ and consequently $x \in s_2(R)$. Hence, $(s_2(R) : R) \subseteq s_2(R)$ and we are done. □

Remark 5.8. If R is a zero symmetric near-ring then $\mathcal{S}({}_RR) = s_2(R)$ if and only if for $x \in R$, $xR \subseteq \mathcal{S}({}_RR)$ implies $x \in \mathcal{S}({}_RR)$.

Proof. Suppose $xR \subseteq \mathcal{S}({}_RR)$ implies $x \in \mathcal{S}({}_RR)$. Since $\mathcal{S}({}_RR) \subseteq s_2(R)$ for any near-ring, we only have to show $s_2(R) \subseteq \mathcal{S}({}_RR)$. Let $y \in s_2(R)$. From Proposition 5.3 we have $yR \subseteq \mathcal{S}({}_RR)$ and from our assumption, it follows that $y \in \mathcal{S}({}_RR)$. Hence $\mathcal{S}({}_RR) = s_2(R)$. The converse is clear from Proposition 5.3 □

6 Special classes of near-rings

Unless otherwise stated, all our near-rings will be zero-symmetric right near-rings. Throughout, $\mathcal{W}$ will denote a universal class of near-rings, that is a class which is closed under taking homomorphic images and ideals. Any subclass of $\mathcal{W}$ will be abstract i.e., it will be isomorphically closed and it will always contain the trivial near-ring. A mapping $\rho : \mathcal{W} \to \mathcal{W}$ is called an ideal mapping if $\rho(R)$ is an ideal of R for every $R \in \mathcal{W}$. An ideal mapping ρ is said to be a Hoehnke radical map (**H-radical** map for short) if:

(H 1) for every $R \in \mathcal{W}$ and for every homomorphism f on R,
$f(\rho(R)) \subseteq \rho(f(R))$;

(H 2) $\rho\left({}^{R}/\rho(R)\right) = 0$ for every $R \in \mathcal{W}$.

The following relationships between the radicals of a near-ring and its ideals play an important role in the general theory of radicals:

An H-radical ρ is:

(H 3) *complete* if $\rho(I) = I \triangleleft R$ implies $I \subseteq \rho(R)$;

(H 4) *idempotent* if $\rho(\rho(R)) = \rho(R)$.

If ρ is an H-radical which is idempotent and complete, then it is called a *Kurosh-Amitsur (KA) radical* map.

Unfortunately, when directly extending various KA-radicals from rings to near-rings, we obtain a radical which is not only "bad" but is also "ugly" in the sense that we may lose one or both conditions (H_3) and (H_4) of the definition of a KA-radical. Many near-ring "radicals" are H-radicals but not necessarily KA-radicals. This leads to two basic paths one can take to obtain "nice" radicals (i.e., KA-radicals) for near-rings. First, one could add more properties to the radical in question so that it is a KA-radical on the class of near-rings and still coincides with its ring theoretic ancestor on the class of rings. The second path we can take (the one we will take in what follows) is to use the direct near-ring analogue of a ring radical but restrict the class of near-rings to which it is applied.

The majority of the so-called radicals of near-rings are defined not as radical classes but as mappings. Unfortunately these classes, as usual, are homomorphically closed but they are not necessarily closed under ideals i.e., the class is not a universal class and, therefore, the traditional theory of radicals cannot be applied here. To handle this situation we have the following generalization of the concept of a radical introduced by Kaarli in [16].

Definition 6.1. Let σ be a mapping which assigns to the near-ring R an ideal $\sigma(R)$ and let $\mathcal{T}$ be a homomorphically closed class of near-rings. The mapping σ is called a $\mathcal{T}-$radical map if:

(a) σ satisfies (H1) and (H2);

(b) σ satisfies (H3) and (H4) for all $R \in \mathcal{T}$.

If in the above definition $\mathcal{T}$ is the class of all near-rings then the concepts $\mathcal{T}-$radical and KA-radical are the same. From [12] we have :

Definition 6.2. A class $\mathcal{F}$ of near-rings is $\mathcal{T}-$special if:

R1. All near-rings from $\mathcal{F}$ are $2-$prime;

R2. $R \in \mathcal{T} \cap \mathcal{F}$ and $A \triangleleft R$ implies $A \in \mathcal{F}$;

R3. $I \triangleleft J \triangleleft N$ and $J/I \in \mathcal{F}$ implies $I \triangleleft R$;

R4. If I is an essential ideal of R $(I \triangleleft \cdot R)$ and $I \in \mathcal{F}$, then $R \in \mathcal{F}$ (i.e. $\mathcal{F}$ is closed under essential extensions).

The mapping σ is called a $\mathcal{T}-$special *radical* radical map if $\sigma(R) = \cap\{I : R/I \in \mathcal{F}\}$ where $\mathcal{F}$ is a $\mathcal{T}-$special class of near-rings. This definition extends the concept of a special radical for rings [7] and [18] to near-rings.

In order to the define a special class of near-ring modules, we recall the following

Let R be a near-ring and $I \triangleleft R$. Let $r \in R$ and $m \in M$. *If M is an R/I-module, then with respect to $rm = (r+I)m$, M becomes an $R-$*module *and $I \subseteq (0 : M)_R$. If M is an $R-$*module *and $I \subseteq (0 : M)_R$, then M is an R/I-module with respect to $(r+I)m = rm$. In both cases, we have that $(0 : M)_{R/I} = (0 : M)_R/I$.*

Now let $\mathcal{T}$ be a nonempty class of zero-symmetric right near-rings which is closed under homomorphic images. For each near-ring R, let $\mathcal{M}_R$ be a class of $R-$modules (possibly empty). Let $\mathcal{M} = \cup\{\mathcal{M}_R : R \text{ is a near-ring}\}$. Then we introduce the notion of a $\mathcal{T}-$special class of near-ring modules:

Definition 6.3. A class $\mathcal{M} = \cup\{\mathcal{M}_R : R \text{ is a near-ring}\}$ of near-ring modules is called a $\mathcal{T}-$special **class** if it satisfies the following conditions:

(M1) *If $M \in \mathcal{M}_R$ and $I \triangleleft R$ with $IM = 0$, then $M \in \mathcal{M}_{R/I}$;*

(M2) *If $I \triangleleft R$* and *$M \in \mathcal{M}_{R/I}$, then $M \in \mathcal{M}_R$;*

(M3) *If $M \in \mathcal{M}_R$ and $I \triangleleft R \in \mathcal{T}$ with $IM \neq 0$, then $M \in \mathcal{M}_I$;*

(M4) *If $M \in \mathcal{M}_R$, then $RM \neq 0$ and $R/(0 : M)_R$ is a $2-$*prime *near-ring;*

(M5) *If $I \triangleleft R \in \mathcal{T}$ and $M \in \mathcal{M}_I$, then there exists an $R-$*module *$N \in \mathcal{M}_R$ such that $(0 : N)_I \subseteq (0 : M)_I$;*

(M6) *If $K \triangleleft I \triangleleft R \in \mathcal{T}$ and there exists a faithful I/K-module $M \in \mathcal{M}_{I/K}$, then $K \triangleleft R$.*

In the previous section, we have seen that there were numerous relationships between a near-ring and its modules. In particular, prime R-ideals of the $R-$module M led to prime ideals of R and, under certain conditions, the converses are also true. It is, therefore, natural to assume that there is a relationship between special radicals of near-rings and special radicals of their modules. In the two theorems that follow, we show the construction of a special class of near-rings from a special class of near-ring modules and the reversal of the process.

From [13] we have the following:

Theorem 6.4. *Let $\mathcal{M} = \cup\{\mathcal{M}_R : R$ is a near-ring$\}$ be a $\mathcal{T}-$special class of near-ring modules. Then*

$\mathcal{F} = \{R :$ there exists $M \in \mathcal{M}_R$ with $(0 : M)_R = 0\} \cup \{0\}$ is a $\mathcal{T}-$special class of near-rings.

Theorem 6.5. *Let $\mathcal{F}$ be a $\mathcal{T}-$special class of near-rings and for the near-ring R, let $\mathcal{M}_R = \{M : M$ is an $R-$module, $RM \neq 0$ and $R/(0 : M)_R \in \mathcal{F}\}$. If $\mathcal{M} = \cup\{\mathcal{M}_R : R$ is a near-ring$\}$ then $\mathcal{M}$ is a $\mathcal{T}-$special class of near-ring modules.*

Proposition 6.1. *Let $\mathcal{M}$ be a $\mathcal{T}-$special class of near-ring modules and suppose $I \lhd R \in \mathcal{R}_0$, where $\mathcal{R}_0$ denotes the class of zero-symmetric near-rings. Let $\mathcal{F}$ be the corresponding $\mathcal{T}-$special class of near-rings. Then $R/I \in \mathcal{F}$ if and only if $I = (0 : M)_R$ for some $M \in \mathcal{M}_R$.*

We now have the following:

Let $\mathcal{K}$ be a $\mathcal{T}-$special class of modules, let $\mathcal{M}_K$ be the class of near-rings defined by $\mathcal{M}_K := \{R :$ there exists $M \in \mathcal{K}_R$ with $(0 : M)_R = 0\}$. Then, $\mathcal{M}_K$ is a $\mathcal{T}-$special class of near-rings and if $\mathcal{R}$ is the corresponding radical then, $\mathcal{R}(R) = \cap\{(0 : M)_R : M \in \mathcal{K}_R\}$ for each near-ring R. Conversely, if $\mathcal{F}$ is a $\mathcal{T}-$special class of near-rings, let $\mathcal{M}_R := \{M$ is an $R-$module, $RM \neq 0$ and $R/(0 : M)_R \in \mathcal{F}\}$ for each near-ring R. Then $\mathcal{M} := \cup\{\mathcal{M}_R\}$ is a $\mathcal{T}-$special class of modules and $r(M) = \cap\{S \lhd_R M : M/S \in \mathcal{M}\}$.

In 1958 Andrunakievich [2] proved the following lemma for associative rings:

"If I is an ideal of a ring R and K is an ideal of I, then $\langle K\rangle_R^3 \subseteq K$ where $\langle K\rangle_R$ is the ideal of R generated by K."

This result has had far reaching consequences through its application to radical theory of associative rings. Because in general the Andrunakievich Lemma is not satisfied for near-rings, the notion of $\mathcal{A}$-near-rings were introduced in [6]. To get the best possible results about the prime radicals which are not KA-radicals, we shall make use of the concept of an $\mathcal{A}$-near-ring. An ideal I of a near-ring R is called an $\mathcal{A}$-*ideal* if for each ideal K of the near-ring I there is some $n \geq 1$, perhaps depending on K, such that $(\langle K\rangle_R)^n \subseteq K$. R is called an $\mathcal{A}$-*near-ring* if every ideal of R is an $\mathcal{A}$-ideal. The class $\mathcal{A}$ is wide and varied, including all distributively generated near-rings and all near-rings which are neither nilpotent nor strongly regular. These and many other examples and the basic properties of $\mathcal{A}$-near-rings are given in [6].

Proposition 6.2. *Let R be any near-ring and $\mathcal{M}_R := \{M : M$ is a prime R-module$\}$. If $\mathcal{M} = \cup\mathcal{M}_R$, then $\mathcal{M}$ is an $\mathcal{A}-$special class of R-modules.*

Proof.

M1 Let $M \in \mathcal{M}_R$ and $I \triangleleft R$ with $IM = 0$. Now M is an R/I-module. We show $M \in \mathcal{M}_{R/I}$. Let $A \triangleleft R/I$ and $B \leq_R M$ such that $AB = 0$. Then, $A = L/I$ for some ideal L of R, and hence $(L/I)B = 0$, i.e., we have $LB = 0$. Since M is prime, we have $LM = 0$ or $B = 0$. By the scalar operation in R/I, $LM = (L+I)M = (L/I)M = AM$. So $AM = 0$ or $B = 0$ whence M is a prime R/I module and we have $M \in \mathcal{M}_{R/I}$.

M2 Let $I \triangleleft R$ and $M \in \mathcal{M}_{R/I}$. M is an R-module w.r.t. $rm = (r+I)m$ for $r \in R$ and $m \in M$. Let $A \triangleleft R$ and $B \leq_R M$ such that $AB = 0$. Then $A/I \triangleleft R/I$ and for all $a \in A$, we have $(a+I)B = aB = 0$. Hence $(A/I)B = 0$ and since $M \in \mathcal{M}_{R/I}$, it follows that $(A/I)M = 0$ or $B = 0$. But for all $a \in A$, we have $(a+I)M = aM$. Hence $AM = 0$ or $B = 0$ and therefore $M \in \mathcal{M}_R$.

M3 Let $M \in \mathcal{M}_R$ and $A \triangleleft R \in \mathcal{A}$ with $AM \neq 0$. Let $B \triangleleft A \triangleleft R$ and $N \leq_A M$ such $BN = 0$. Since $R \in \mathcal{A}$, there exists $n \in \mathbb{N}$ such that $\langle B\rangle_R^n N \subseteq BN = 0$. Let m be minimal such that $\langle B\rangle_R^m N \subseteq BN = 0$. If $m = 1$, then $\langle B\rangle_R N = 0$. If N is also an $R-$submodule then, since M is a prime module, we have $\langle B\rangle_R M = 0$ or $N = 0$. If $N = 0$ then we are done. Suppose $N \neq 0$. Now $\langle B\rangle_R M = 0$ and we have $BM = 0$. If N is not an $R-$ submodule of M, then there exists $t \in N$ such that $Rt \subsetneq N$. Rt is a nonzero $R-$ submodule of M. Now we have $\langle B\rangle_R Rt \subseteq \langle B\rangle_R t \subseteq \langle B\rangle_R N \subseteq BN = 0$. Since M is a prime $R-$module and Rt is a nonzero R submodule we have $BM \subseteq \langle B\rangle_R M = 0$. If $m > 1$, then $\langle B\rangle_R^{n-1} N \neq 0$ and there exists $x \in \langle B\rangle_R^{m-2} N \subseteq M$ such that $\langle B\rangle_R x \neq 0$. Now, $\langle B\rangle_R \langle B\rangle_R x = 0$. Since $\langle B\rangle_R x$ is a nonzero R submodule, and $M \in \mathcal{M}_R$ we have $BM = 0$ and we are done.

M4 Let $M \in \mathcal{M}_R$. Hence $RM \neq 0$. Since $(0 : M)_R$ is a $2-$prime ideal of R, $R/(0 : M)_R$ is a $2-$prime near-ring.

M5 Let $R \in \mathcal{A}$ with $I \triangleleft R$ such that $M \in \mathcal{M}_I$. Since $M \in \mathcal{M}_I$, $(0 : M)_I$ is a $2-$prime ideal of I. So $(0 : M)_I \triangleleft I \triangleleft R$. Since $R \in \mathcal{A}$ and $I/(0 : M)_I\rangle$a $2-$prime near-ring, it follows from [6, Lemma 1] that $(0 : M)_I \triangleleft R$. Now choose $K/(0 : M)_I$ to be the ideal of $R/(0 : M)_I$ which is maximal with respect to $I/(0 : M)_I \cap K/(0 : M)_I = 0$. Then it is well known that $I/(0 : M)_I \cong I/K \triangleleft\cdot\, R/K$. Since $I/(0 : M)_I$ is a $2-$prime near-ring and since the class of 2−prime near-rings is essentially closed it follows that R/K is a 2−prime near-ring. Now R/K is an R−module. We show that $H = R/K$ is the required R−module. Clearly, $R(R/K) \neq 0$. We show that $(0 : R/K)_R = K$. So let $x \in K$. Then $x(r + K) = xr + K = K$ for all $r \in R$. Therefore $x \in (0 : R/K)_R$. Conversely, let $x \in (0 : R/K)_R$. Then $xR \subseteq K$. Since R/K is a 2−prime near-ring, K is a 2−prime ideal of R. But $xR \subseteq K$ and K is 2−prime implies that $x \in K$. Hence we have that $(0 : R/K)_R = K$. Now $R/(0 : R/K)_R = R/K$and $R(R/K) \neq 0$. Hence $H = R/K \in \mathcal{M}_R$. Finally, we show that $(0 : R/K)_I \subseteq (0 : M)_I$. Let $x \in (0 : R/K)_I$. Since $I \lhd R$, we have that $xR \subseteq I$. Furthermore, $x(R/K) = 0 \Longrightarrow xR \subseteq K$. Hence $xR \subseteq I \cap K$, and from the definition of $K/(0 : M)_I$, we get $xR \subseteq I \cap K \subseteq (0 : M)_I$. Hence $xRM = 0$. Now $xIM \subseteq xRM = 0$ implies $xI \subseteq (0 : M)_I$. Since $I/(0 : M)_I$ is a 2−prime near-ring, $(0 : M)_I$ is a 2−prime ideal of I and we get $x \in (0 : M)_I$. So $(0 : R/K)_I \subseteq (0 : M)_I$ and (**M5**) is satisfied.

M6 Let $K \lhd I \lhd R \in \mathcal{A}$ and $M \in \mathcal{M}_{I/K}$ be a faithful I/K-module. Since $M \in \mathcal{M}_{I/K}$ and M is faithful, we have that $I/K = \dfrac{I/K}{(0 : M)_{I/K}}$ is a 2−prime near-ring. Thus K is a 2 -prime ideal of I. Since I is an $\mathcal{A}$−ideal of R it follows from [6, Lemma 1] that $K \lhd R$. □

Remark 6.6. If $\mathcal{M}_p$ denotes the $\mathcal{A}$−special class of prime near-ring modules, then the $\mathcal{A}$−special radical induced by $\mathcal{M}_p$ on a near-ring R is given by:

$$\begin{aligned}\wp_2(R) &= \cap\{(0 : M)_R : M \text{ is a prime } R-\text{ module }\}\\ &= \cap\{I \lhd R : I \text{ a 2−prime ideal }\}\end{aligned}$$

Let R be an $\mathcal{A}$ -near-ring and let $\mathcal{M}_s = \{M : M \text{ is an} s\text{−prime } R - \text{module}\}$. We want to show that $\mathcal{M}_s$ is an $\mathcal{A}$−special class of near-ring modules. Since we already know that the class of prime modules is an $\mathcal{A}$−special class, it follows from Proposition 4.1 that we only have to show that conditions (M1) to (M6) of Definition 6.3 are satisfied for condition (b) of Corollary 4.5

Proposition 6.3. *Let R be any $\mathcal{A}$−near-ring and $\mathcal{M}_R := \{M{:}M$ is an s−prime R-module$\}$. If $\mathcal{M}_s = \cup\mathcal{M}_R$, then $\mathcal{M}_s$ is a $\mathcal{A}$−special class of near-ring modules.*

Proof.

M1 Let $M \in \mathcal{M}_R$ and $I \lhd R$ with $IM = 0$. Now $M \in \mathcal{M}_R$ implies that $R/(0 : M)_R$ contains no nonzero nil ideals. But $(R/I)/(0 : M)_{R/I} = (R/I)/([(0 : M)_R]/I) \simeq R/(0 : M)_R$. Hence $(R/I)/(0 : M)_{R/I}$ contains no nonzero nil ideals and thus we have $M \in \mathcal{M}_{R/I}$.

M2 If $I \lhd R$ and $M \in \mathcal{M}_{R/I}$, $(R/I)/(0 : M)_{R/I}$ contains no nonzero nil ideals. So $R/(0 : M)_R \simeq (R/I)/(0 : M)_{R/I}$ has no nonzero nil ideals implies $M \in \mathcal{M}_R$.

M3 Let $M \in \mathcal{M}_R$ and $I \lhd R \in \mathcal{A}$ with $IM \neq 0$. Then $R/(0:M)_R$ contains no nonzero nil ideals. Now $I/(0:M)_I = I/[(0:M)_R \cap I] \simeq (I+(0:M)_R)/(0:M)_R \lhd R/(0:M)_R$. Since R is an $\mathcal{A}-$near-ring it follows from [6, Corollary 12] that $I/(0:M)_I$ also contains no nonzero nil ideals. Hence $M \in \mathcal{M}_I$.

M4 Let $M \in \mathcal{M}_R$. Hence $RM \neq 0$. Since $(0:M)_R$ is a $2-$prime ideal of R, $R/(0:M)_R$ is a $2-$prime near-ring.

M5 Let $R \in \mathcal{A}$ with $I \lhd R$ such that $M \in \mathcal{M}_I$. As in the proof of $M5$ of Proposition 6.2 choose $K/(0:M)_I$ to be the ideal of $R/(0:M)_I$ which is maximal with respect to $I/(0:M)_I \cap K/(0:M)_I = 0$. Then $I/(0:M)_I \cong I/K \lhd \cdot R/K$. Since $M \in \mathcal{M}_I$, $I/(0:M)_I$ contains no nonzero nil ideals. Hence we also have that R/K contains no nonzero nil ideals. But we know that $K = (0:R/K)_R$. Hence $R/(0:R/K)_R$ contains no nonzero nil ideals implying that $R/K \in \mathcal{M}_R$.

M6 Let $K \lhd I \lhd R \in \mathcal{A}$ and $M \in \mathcal{M}_{I/K}$ be a faithful I/K-module. Since M is faithful I/K-module, $(0:M)_{I/K} = 0$. Since $M \in \mathcal{M}_{I/K}$we have that $(0:M)_{I/K} = 0$ is a $2-$ $s-$prime ideal of I/K and consequently I/K is a $2-$ $s-$prime near-ring. From [6, Lemma 1] we get $K \lhd R$. □

Proposition 6.4. *If $\mathcal{M}_s$ is a $\mathcal{A}-$special class of near-ring modules, then the $\mathcal{A}-$special radical induced by $\mathcal{M}_s$ on a near-ring R is given by:*

$$\begin{aligned} s_2(R) &= \cap\{(0:M)_R : M \quad \text{is an } s\text{-prime } R-\text{module}\} \\ &= \cap\{I \lhd R : I \ \text{a } 2-s\text{-prime ideal }\} \end{aligned}$$

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