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ALGORITHMIC SCHEMES FOR THE MULTIPLE-SETS SPLIT EQUALITY PROBLEM

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Abstract

In this paper, for solving the multiple-sets split equality problem (MSSEP), we give a general approach to construct iterative methods. We present an weakly convergent string-averaging algorithmic scheme and its relaxed variant, that contain the cyclic and simultaneous iterative methods as particular cases. Then, we also propose a combination of the steepest-descent method with one of these scheme to obtain strong convergence. In our methods, we do not need to have any information on the operator norms. We also give numerical examples for illustrating our main methods.

1. Introduction

Let H_1, H_2 and H_3 be real Hilbert spaces. Let J_1 and J_2 be two index sets with N and M elements, respectively, where N and M are any positive integers. Let $\{C_i\}_{i\in J_1}$ and $\{Q_j\}_{j\in J_2}$ be two families of closed convex subsets in H_1 and H_2, respectively, and let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear mappings with the standard norms $\|A\|$ and $\|B\|$, respectively. We denote by I, $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the identity mapping, inner product and norm for any Hilbert space.

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The multiple-sets split equality problem is to find a point $z_* = [x_*, y_*]$ with the properties:

$$x_* \in C := \cap_{i\in J_1} C_i \quad \text{and} \quad y_* \in Q := \cap_{j\in J_2} Q_j \quad \text{such that} \quad Ax_* = By_*. \tag{1.1}$$

Denote the set of solutions for (1.1) by Γ, assumed to be non-empty in this paper.

Clearly, when $H_2 = H_3$ and $B = I$, the MSSEP reduces to the *multiple-sets split feasibility problem* (MSSFP), that was first introduced by Censor and Elfving [1] for modeling inverse problems that arise from phase retrievals and in image reconstruction [2]. Recently, the MSSFP can also be used to model the intensity-modulated radiation therapy [3-6] and references therein.

In the case that $N = M = 1$, the MSSEP reduces to the split equality problem (SEP), that is to find points x_* and y_* such that

$$x_* \in C, \quad y_* \in Q \quad \text{and} \quad Ax_* = By_*. \tag{1.2}$$

Problem (1.2) was introduced and studied by Byrne and Moudafi [7] in finite-dimensional spaces. This is actually an optimization problem with weak coupling in the constraint and its interest covers many situations, for instance, in domain decomposition for PDEs [8] and game theory [9]. In order to solve problem (1.2), they introduced the weak convergent CQ-like method, $z^1 = [x^1, y^1] \in C \times Q$ and

$$\begin{aligned} x^{k+1} &= P_C(x^k - \gamma_k A^*(Ax^k - By^k)),\\ y^{k+1} &= P_Q(x^k + \gamma_k B^*(Ax^k - By^k)),\ \forall k \geq 1, \end{aligned} \tag{1.3}$$

where A^* and B^* are the adjoints of A and B, respectively, and $\gamma_k = \gamma$ is chosen in the interval $(a, b) \subset (0, \min\{1/\|A\|^2, 1/\|B\|^2\})$ for all $k \geq 1$. So, the choice value γ depends on the norms $\|A\|$ and $\|B\|$, that are not easy to be calculated in practice. To overcome the difficulty, Dong et al. [10] and Vuong et al. [11] indicated that γ_k can be chosen by

$$\gamma_k = \frac{\rho_k f(x^k, y^k)}{a_k} \quad \text{with } \rho_k \in (0, 4), \tag{1.4}$$

where $f(x, y) = \|Ax - By\|^2/2$ and $a_k = \|A^*(Ax^k - By^k)\|^2 + \|B^*(Ax^k - By^k)\|^2$. Next, Chuang and Du [12] proved weak convergence for method (1.3) when γ_k is chosen in the interval $(0, 2/(\|A\|^2+\|B\|^2))$ such that $\liminf_{k\to\infty} \gamma_k(2-\gamma_k(\|A\|^2 + \|B\|^2)) > 0$ with an additional conditions on (x^k, y^k). Recently, Wang [13] gave a new way to select the parameter γ_k. The iterative regularization method and several projection methods have been investigated in [14-19].

Clearly, in the Hilbert space $H = H_1 \times H_2$ with an inner product and a norm denoted and defined by $\langle z^1, z^2\rangle = \langle x^1, x^2\rangle + \langle y^1, y^2\rangle$ and $\|z\| = (\|x\|^2+\|y\|^2)^{1/2}$, respectively, where $z = [x, y]$ and $z^i = \langle x^i, y^i \rangle$ with $x, x^i \in H_1$ and $y, y^i \in H_2$ for $i = 1, 2$, method (1.3) can be re-written in the compact form,

$$z^{k+1} = P_S(I - \gamma G^* G) z^k, \ z^1 \in H, \tag{1.5}$$

where $G = [A, -B]^T : H \to H$ and $S = C \times Q$. Further, Li and Chen [20] extended (1.5) to MSSEP (1.1) with $N > M$ by a sequential iterative method,

$$z^{k+1} = P_{S_{m(k)}}(I - \gamma G^* G) z^k, \tag{1.6}$$

where $m(k) = k \mod (N + 1)$ with $Q_j = H_2$, for $M < j \leq N$, is some additional set and $S_i = C_i \times Q_i$ for $i = 1, \cdots, N$, and a simultaneous one,

$$z^{k+1} = \sum_{i=1}^{N} \lambda_i P_{S_i}(I - \gamma G^* G) z^k, \tag{1.7}$$

where $\lambda_i > 0$ for all i such that $\sum_{i=1}^{N} \lambda_i = 1$ and $\gamma \in (0, 2/\|G\|^2)$. They proposed also several iterative methods of Krasnoselskii-Mann's type

$$\begin{aligned} z^{k+1} &= (1 - t_k) z^k + t_k P_{S_N}(I - \gamma G^* G) \cdots P_{S_1}(I - \gamma G^* G) z^k, \\ z^{k+1} &= (1 - t_k) z^k + t_k \sum_{i=1}^{N} \lambda_i P_{S_i}(z^k - \gamma G^* G z^k), \end{aligned} \tag{1.8}$$

with a condition on $t_k : \sum_{k=1}^{\infty} t_k(1 - t_k) = \infty$. All methods (1.5)-(1.8) converge weakly to a point in Γ. In order to obtain strong convergence for the sequence $\{z^k\}$, defined by (1.6) or (1.7), they also introduced several iterative methods of Halpern's type, $\overline{z} = [\overline{u}, \overline{v}] \in H$ and

$$\begin{aligned} z^{k+1} &= t_k \overline{z} + (1 - t_k) P_{S_{m(k)}}(z^k - \gamma G^* G z^k), \\ z^{k+1} &= t_k \overline{z} + (1 - t_k) P_{S_N}(I - \gamma G^* G) \cdots P_{S_1}(I - \gamma G^* G) z^k, \\ z^{k+1} &= t_k \overline{z} + (1 - t_k) \sum_{i=1}^{N} \lambda_i P_{S_i}(z^k - \gamma G^* G z^k), \end{aligned} \tag{1.9}$$

with a new condition on t_k, that is
(t) $t_k \in (0, 1)$ for all $k \geq 1$, $\lim_{k\to\infty} t_k = 0$, $\sum_{k=1}^{\infty} t_k = \infty$ and
(t') either $\sum_{k=1}^{\infty} |t_{k+1} - t_k| < \infty$ or $\lim_{k\to\infty}(t_k/t_{k+1}) = 1$.
Further, Zhao and Shi [18] introduced a new extragradient-type method for the MSSEP. Meantime, Tian et al. [17] proposed a new iterative method, in which the iterative step size is split self-adaptive without needing to have any information about $\|A\|$ and $\|B\|$.

When $H_1 = H_2 = H_3$ and $A = B = I$, problem (1.1) reduces to the convex feasibility problem, that is to find a point $p_* \in \cap_{i=1}^{n} C_i$ where n is a positive integer and C_i is a closed convex set in a Hilbert space H for all $1 \le i \le n$. To solve the convex feasibility problem, Censor et al [21] introduced a *string-averaging* algorithmic scheme, that projects a point sequentially along several independent strings of constraints. Projecting along each string is sequential, but the strings are independent and projecting along them can be performed in parallel. In final, the end-points of strings of sequential projections onto the constraints are averaged.

The purpose of this paper is to use the results listed above to design a general scheme for iterative methods, solving (1.1). The rest of this paper is organized as follows. In Section 2, we list some related facts, that will be used in the proof of our results. In Section 3, we propose a string-averaging scheme to solve (1.1) and show its weak convergence. A relaxed string-averaging scheme is considered in Section 4. In order to obtain strong convergence, we give a combination of the string-averaging scheme with the steepest-descent method for monotone mappings in Section 5. Finally, in Section 6, we give numerical experiments for illustrating our main results.

2. Preliminaries

In any real Hilbert space H, we have the following inequality,

$$\|u+v\|^2 \le \|u\|^2 + 2\langle v, u+v\rangle,\ \forall u, v \in H.$$

Definitions 2.1 A mapping T from a subset Ω of H into H is called:

(i) nonexpansive, if $\|Tu - Tv\| \le \|u - v\|$ for all $u, v \in \Omega$;
(ii) contractive, if $\|Tu - Tv\| \le \tilde{a}\|u - v\|$ for a fixed $\tilde{a} \in [0, 1)$ and for all $u, v \in \Omega$;
(iii) γ-inverse strongly monotone, if $\gamma\|Tu - Tv\|^2 \le \langle Tu - Tv, u - v\rangle$ for all $u, v \in \Omega$, where γ is a positive number;
(iv) firmly nonexpansive, if there holds (iii) with $\gamma = 1$.
(v) η-strongly monotone and γ-strictly pseudocontractive mapping, if there hold, respectively,

$$\begin{aligned}&\langle Tx_1 - Tx_2, x_1 - x_2\rangle \ge \eta\|x_1 - x_2\|^2 \quad \text{and}\\ &\langle Tx_1 - Tx_2, x_1 - x_2\rangle \le \|x_1 - x_2\|^2 - \gamma\|(I - T)x_1 - (I - T)x_2\|^2\end{aligned}$$

for all $x_1, x_2 \in \Omega$, where η and γ are some positive real numbers.

For a closed convex subset Ω of H, there exists a mapping $P_\Omega : H$ onto Ω such that $P_\Omega(u) = \inf_{v\in\Omega} \|v - u\|$ for each $u \in H$. The mapping P_Ω is called the metric projection onto Ω. We know that P_Ω is firmly nonexpansive (hence, nonexpansive); $I - P_\Omega$ is also firmly nonexpansive; $\langle P_\Omega u - z, u - P_\Omega u\rangle \ge 0,\ u \in H, z \in \Omega$; and for any $u \in H, z \in \Omega$ we have that $\|u - P_\Omega u\|^2 + \|P_\Omega u - z\|^2 \le$

$\|u - z\|^2,\ u \in H, z \in \Omega$. The set of fixed points for T from Ω into H is denoted by $\mathrm{Fix}(T)$, i.e., $\mathrm{Fix}(T) := \{u \in \Omega : Tu = u\}$.

Lemma 2.1 (see, [22]) *Let Ω be a closed convex subset of a real Hilbert space H and let $T : \Omega \to \Omega$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. If $\{u^k\}$ is a sequence in Ω weakly converging to u and if $(I - T)u^k$ converges strongly to v, then $(I - T)u = v$. In particular, if $v = 0$, then $u \in \mathrm{Fix}(T)$.*

Lemma 2.2 (see, [23]) *Let $\{a_k\}$, $\{t_k\}$ and $\{c_k\}$ be sequences of real numbers such that, for all $k \geq 1$,*
(i) $a_{k+1} \leq (1 - t_k)a_k + t_k c_k$;
(ii) $a_k \geq 0$;
(iii) *There holds condition* (t);
(iv) $\limsup_{k\to\infty} c_k \leq 0$;
Then, $\lim_{k\to\infty} a_k = 0$.

Lemma 2.3 (see, [24]) *Let $\{a_k\}$ be a sequence of real numbers such that there exists a subsequence $\{k_l\}$ of $\{k\}$ such that $a_{k_l} < a_{k_l+1}$ for all positive integer l. Then, there exists a nondecreasing sequence $\{m_k\}$ of positive integers such that $m_k \to \infty, a_{m_k} \leq a_{m_k+1}$ and $a_k \leq a_{m_k+1}$ for all (sufficiently large) $k \geq 1$. In fact, $m_k = \max\{j \leq k : a_j \leq a_{j+1}\}$.*

Lemma 2.4 (see, [25]) *Let H be a real Hilbert space and $\{z^k\}$ a sequence in H such that there exists a nonempty closed set $\Omega \subseteq H$ satisfying $\omega_\omega(z^k) \subset \Omega$ and $\lim_{k\to\infty} \|z^k - z\|$ exists for every $z \in \Omega$. Then there exists $\tilde{z} \in \Omega$ such that $\{z^k\}$ converges weakly to $\tilde{z}$.*

Lemma 2.5 (*see,* [26]) *Let H be a real Hilbert space and let $F : H \to H$ be an η-strongly monotone and γ-strictly pseudocontractive mapping with $\eta + \gamma > 1$. Then, for any $t \in (0, 1), I - tF$ is contractive with constant $1 - t\tau$ where $\tau = 1 - \sqrt{(1 - \eta)/\gamma}$.*

3. A string-averaging scheme for the MSSEP

Let the *string* $J_1^t = (i_1^t, i_2^t, \cdots, i_{\gamma(J_1^t)}^t)$ be a finite nonempty subset of J_1, for every $t = 1, 2, \cdots, S_1$, where the length of the string J_1^t, denoted by $\gamma(J_1^t)$, is the number of elements in J_1^t. Put $T_1^t := P_{i_{\gamma(J_1^t)}^t} \cdots P_{i_2^t} P_{i_1^t}$, where $P_{i_l^t} = P_{C_{i_l^t}}$, for $l = 1, 2, \cdots, \gamma(J_1^t)$ and $t = 1, 2, \cdots, S_1$. Given a positive weight vector $\beta = (\beta_1, \beta_2, \cdots, \beta_{S_1})$ with $\sum_{t=1}^{S_1} \beta_t = 1$, we define the algorithmic mapping $\mathcal{P}_1 := \sum_{t=1}^{S_1} \beta_t T_1^t$. We suppose that every element of J_1 appears in at least one of the strings J_1^t. Analogously, for the family $\{Q_j\}_{j\in J_2}$, we can construct the mapping $\mathcal{P}_2 := \sum_{t=1}^{S_2} \eta_t T_2^t$ where $T_2^t := P_{j_{\gamma(J_2^t)}^t} \cdots P_{j_2^t} P_{j_1^t}$, $P_{j_l^t} = P_{Q_{j_l^t}}$ for $t = 1, 2, \cdots, S_2$, $l = 1, 2, \cdots, \gamma(J_2^t)$ and $\eta = (\eta_1, \eta_2, \cdots, \eta_{S_2})$ is also a positive weight vector such that $\sum_{t=1}^{S_2} \eta_t = 1$.

First, we need to prove the following lemma.

Lemma 3.1 $z = [u,v] \in \Gamma$ if *and only if* $(I-\mathcal{P}_1)u = (I-\mathcal{P}_2)v = 0$ *and* $Au = Bv$.

Proof. Clearly, when $z = [u,v] \in \Gamma$, we have that $Au = Bv$, $u \in C_i$ and $v \in Q_j$ for every $i \in J_1$ and $j \in J_2$. Consequently, for all $t = 1, 2, \cdots, S_1$ we have that $T_1^t u = u$ and for $t = 1, 2, \cdots, S_2$, $T_2^t v = v$. Consequently, $(I-\mathcal{P}_1)u = (I-\mathcal{P}_2)v = 0$. Inversely, we have to prove that if $z = [u,v]$ satisfies the equalities then $z \in \Gamma$. Take any point $[p,q] \in \Gamma$. It is easy to see that

$$\begin{aligned}\|u-p\|^2 = \|\mathcal{P}_1 u - p\|^2 &\le \sum_{t=1}^{S_1} \beta_t \|T_1^t u - p\|^2 \\ &\le \|u-p\|^2 - \sum_{t=1}^{S_1} \beta_t \sum_{l=1}^{\gamma(I_t)} \|U^{i_l^t} u - U^{i_{l-1}^t} u\|^2,\end{aligned}$$

and hence, $\|U^{i_l^t}u - U^{i_{l-1}^t}u\|^2 = 0$ for $l = 1, 2, \cdots, \gamma(J_1^t)$, where $U^{i_l^t} = P_{i_l^t} \cdots P_{i_2^t} P_{i_1^t}$ and $U^{i_0^t} = I$. Taking $l = 1$, we obtain that $U^{i_1^t}u = u$, which together with the case that $l = 2$ implies $U^{i_2^t}u = u$. Repeating the process for $l = 3, \cdots, \gamma(J_1^t)$, we get that $U^{i_l^t}u = u$ for $l = 3, \cdots, \gamma(J_1^t)$. Finally, $U^{i_l^t}u = u$ for $l = 1, 2, \cdots, \gamma(J_1^t)$ and $t = 1, 2, \cdots, S_1$. Since each element of J_1 appears in at least one J_1^t, $P_{C_i}u = u$ for each $i \in J_1$. By the similar argument, we get that $P_{Q_j}v = v$ for each $j \in J_2$. From the last two equalities and $Au = Bv$ it follows that $z \in \Gamma$. This completes the proof. □

Now, we consider a string-averaging scheme, $z^1 = [x^1, y^1]$, $x^1 \in H_1$, $y^1 \in H_2$, and

$$\begin{aligned}x^{k+1} &= \mathcal{P}_1(x^k - \gamma_k A^*(Ax^k - By^k)), \\ y^{k+1} &= \mathcal{P}_2(y^k + \gamma_k B^*(Ax^k - By^k)),\end{aligned} \tag{3.1}$$

where γ_k is chosen by

$$\gamma_k = \frac{\rho_k f(x^k, y^k)}{a_k + \varepsilon_k} \tag{3.2}$$

with $\rho_k, f(x,y)$ and a_k defined in (1.4) and an assumption:
(ε): $\{\varepsilon_k\}$ is a bounded sequence of positive real numbers and has $\liminf_{k\to\infty} \varepsilon_k > 0$.

Theorem 3.1 *Let H_1, H_2 and H_3 be real Hilbert spaces, let A and B be two bounded linear mappings from H_1 and H_2 into H_3, respectively, and let C_i and Q_j be two closed convex subsets in H_1 and H_2, respectively, for each $i \in J_1$ and $j \in J_2$. Assume that there holds assumption (ε). Then, the sequence $\{z^k = [x^k, y^k]\}$, defined by* (3.1) *and* (3.2), *as $k \to \infty$, converges weakly to a solution of* (1.1).

Proof. Let $z = [p, q] \in \Gamma$. Then, by Lemma 3.1, $\mathcal{P}_1 p = p$, $\mathcal{P}_2 q = q$ and $Ap = Bq$. Therefore, from (3.1) and the nonexpansivity of $\mathcal{P}_1$ with $\mathcal{P}_2$, we get that

$$\begin{aligned} \|x^{k+1} - p\|^2 &= \|\mathcal{P}_1(x^k - \gamma_k A^*(Ax^k - By^k)) - \mathcal{P}_1 p\|^2 \\ &\le \|x^k - p - \gamma_k A^*(Ax^k - By^k)\|^2 \\ &= \|x^k - p\|^2 - 2\gamma_k \langle A^*(Ax^k - By^k), x^k - p\rangle \\ &\quad + \gamma_k^2 \|A^*(Ax^k - By^k)\|^2 \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \|y^{k+1} - q\|^2 &= \|\mathcal{P}_2(y^k - \gamma_k A^*(Ax^k - By^k)) - \mathcal{P}_2 q\|^2 \\ &\le \|y^k - q + \gamma_k B^*(Ax^k - By^k)\|^2 \\ &= \|y^k - q\|^2 + 2\gamma_k \langle B^*(Ax^k - By^k), y^k - q\rangle \\ &\quad + \gamma_k^2 \|B^*(Ax^k - By^k)\|^2. \end{aligned} \tag{3.4}$$

Since $Ap = Bq$,

$$\begin{gathered} -\langle A^*(Ax^k - By^k), x^k - p\rangle + \langle B^*(Ax^k - By^k), y^k - q\rangle = \\ -\langle Ax^k - By^k, Ax^k - Ap\rangle + \langle Ax^k - By^k, By^k - Bq\rangle = -\|Ax^k - By^k\|^2. \end{gathered} \tag{3.5}$$

Therefore, from (3.3)-(3.5) we have that

$$\|z^{k+1} - z\|^2 \le \|z^k - z\|^2 + \gamma_k^2(a_k + \varepsilon_k) - 4\gamma_k f(x^k, y^k).$$

The last inequality together with (3.2) implies that

$$\|z^{k+1} - z\|^2 \le \|z^k - z\|^2 - \rho_k(4 - \rho_k)\frac{f^2(x^k, y^k)}{a_k + \varepsilon_k}, \tag{3.6}$$

from which and (1.4), we obtain the boundedness of the sequence $\{z^k\}$ in H and the existence of $\lim_{k\to\infty} \|z^k - z\|$. So, there exists $\lim_{k\to\infty} \|x^k - p\|$ and

$$\lim_{k\to\infty} f(x^k, y^k) = 0. \tag{3.7}$$

Next, put

$$\begin{aligned} &u^k := x^k + h_k \text{ where } h_k = -\gamma_k A^*(Ax^k - By^k) \quad \text{and} \\ &v^k := y^k + g_k \text{ where } g_k = \gamma_k B^*(Ax^k - By^k). \end{aligned}$$

It is not difficult to verify that h_k, g_k converge strongly to zero as $k \to \infty$. As in the proof of Lemma 3.1,

$$\begin{aligned}\|x^{k+1}-p\|^2 = \|\mathcal{P}_1 u^k - p\|^2 &\le \sum_{t=1}^{S_1} \beta_t \|T_t^1 u^k - p\|^2 \\ &\le \|u^k - p\|^2 - \sum_{t=1}^{S_1} \beta_t \sum_{l=1}^{\gamma(I_t)} \|U^{i_l^t} u^k - U^{i_{l-1}^t} u^k\|^2 \\ &\le \|x^k - p\|^2 + 2\langle y^k, x^k - p\rangle - \sum_{t=1}^{S_1} \beta_t \sum_{l=1}^{\gamma(J_1^t)} \|U^{i_l^t} u^k - U^{i_{l-1}^t} u^k\|^2,\end{aligned}$$

and hence,

$$\lim_{k\to\infty} \|U^{i_l^t} u^k - U^{i_{l-1}^t} u^k\|^2 = 0 \tag{3.8}$$

for $l = 1, 2, \cdots, \gamma(J_1^t)$, where the mapping $U^{i_l^t} = P_{i_l^t} \cdots P_{i_2^t} P_{i_1^t}$ and $U^{i_0^t} = I$. By taking $l = 1$ in (3.8), we obtain that $\lim_{k\to\infty} \|U^{i_1^t} u^k - u^k\| = 0$, which together with the case that $l = 2$ implies $\lim_{k\to\infty} \|U^{i_2^t} u^k - u^k\| = 0$. Repeating the process for $l = 3, \cdots, \gamma(J_1^t)$, we get that $\lim_{k\to\infty} \|U^{i_l^t} u^k - u^k\| = 0$ for $l = 3, \cdots, \gamma(J_1^t)$. Finally, $\lim_{k\to\infty} \|U^{i_l^t} u^k - u^k\| = 0$ for $l = 1, 2, \cdots, \gamma(J_1^t)$ and $t = 1, 2, \cdots, S_1$. Since each element of J_1 appears in at least one J_1^t, $\lim_{k\to\infty} \|P_{C_i} u^k - u^k\| = 0$ for each $i \in J_1$. Noting $\lim_{k\to\infty} h_k = 0$,

$$\lim_{k\to\infty} \|P_{C_i} x^k - x^k\| = 0 \ \forall i \in J_1. \tag{3.9}$$

Similarly, we get that

$$\lim_{k\to\infty} \|P_{Q_j} y^k - y^k\| = 0 \ \forall j \in J_2. \tag{3.10}$$

Since $\{z^k\}$ is bounded, there exists a subsequence $\{z^{k_m}\}$ of $\{z^k\}$ such that it converges weakly to a point $\tilde{z} = [\tilde{x}, \tilde{y}] \in H$, as $m \to \infty$, where $z^{k_m} = [x^{k_m}, y^{k_m}]$. Then, $\{x^{k_m}\}$ and $\{y^{k_m}\}$ converge weakly to the points $\tilde{x} \in H_1$ and $\tilde{y} \in H_2$, respectively. From Lemma 2.1 with (3.9) and (3.10), we can conclude that $P_{C_i}\tilde{x} = \tilde{x}$ and $P_{Q_j}\tilde{y} = \tilde{y}$ for all $i \in J_1$ and $j \in J_2$. On the other hand, as the function $f(x, y)$ is a convex non-negative function on H, from (3.7) we have that

$$0 \le f(\tilde{x}, \tilde{y}) \le \liminf_{m\to\infty} f(x^{k_m}, y^{k_m}) = \lim_{k\to\infty} f(x^k, y^k) = 0.$$

Consequently, $\|A\tilde{x} - B\tilde{y}\| = 0$. By Lemma 2.1, $\tilde{z} \in \Gamma$. By the similar argument as the above, we can conclude that every weak cluster point of $\{z^k\}$ belongs to Γ. By Lemma 2.4, all the sequence $\{z^k\}$ converges weakly to a point in Γ. Thus, the proof is completed. □

Remarks.

1. Taking $S_1 = S_2 = 1$ with $\gamma(J_1^t) = N$ and $\gamma(J_2^t) = M$, we have the method,

$$x^{k+1} = P_{C_N} \cdots P_{C_1}(x^k - \gamma_k A^*(Ax^k - By^k)),$$
$$y^{k+1} = P_{Q_M} \cdots P_{Q_1}(y^k + \gamma_k B^*(Ax^k - By^k)),$$

that is simpler than (1.6).
2. Taking $S_1 = N$ and $S_2 = M$ with $\gamma(J_1^t) = \gamma(J_2^t) = 1$ for every t, we get the method,

$$x^{k+1} = \sum_{i=1}^{N} \beta_i P_{C_i}(x^k - \gamma_k A^*(Ax^k - By^k)),$$
$$y^{k+1} = \sum_{i=1}^{M} \eta_j P_{Q_j}(y^k + \gamma_k B^*(Ax^k - By^k)).$$

We see that the last method is different from (1.7).

4. A relaxed string-averaging scheme for the MSSEP

In the previous string-averaging scheme, we assume that all the projections P_{C_i} and P_{Q_j} can be easily calculated, but in practice they are sometime difficult to compute or even impossible. In this section, we give a relaxed variant for algorithmic scheme (3.1)-(3.2). First, we assume that the convex subsets C_i and Q_j in this part satisfy the following assumptions:

(a1) The subset C_i for all $i \in J_1$ is given by $C_i = \{x \in H_1 : c_i(x) \le 0\}$, where $c_i : H_1 \longrightarrow (-\infty, +\infty)$ is a convex function.

The subset Q_j for all $j \in J_2$ is given by $Q_j = \{y \in H_2 : q_j(y) \le 0\}$, where $q_j : H_2 \longrightarrow (-\infty, +\infty)$ is a convex function.

(a2) For any $x \in H_1$ and $y \in H_2$, at least one of subdifferetial $\xi_i \in \partial c_i(x)$ and $\theta_j \in \partial q_j(y)$ can be computed, where $\partial c_i(x)$ and $\partial q_j(y)$ are the subdifferentials of $c_i(x)$ and $q_j(y)$ at the points x and y, respectively,

$$\partial c_i(x) = \{\xi_i \in H_1 : c_i(x') \ge c_i(x) + \langle \xi_i, x' - x\rangle \text{ for all } x' \in H_1\},$$
$$\partial q_j(y) = \{\theta_j \in H_2 :\ q_j(y') \ge q_j(y) + \langle \theta_j, y' - y\rangle \text{ for all } y' \in H_2\}.$$

We define the following half-spaces:

$$C_i^k = \{x \in H_1 :\ c_i(x^k) + \langle \xi_i^k, x^k - x\rangle \le 0\},$$

where $\xi_i^k \in \partial c_i(x^k)$ for $i \in J_1$, and

$$Q_j^k = \{y \in H_2 :\ q_j(y^k) + \langle \theta_j^k, y^k - y\rangle \le 0\},$$

where $\theta_j^k \in \partial q_j(y^k)$ for $j \in J_2$.

Put $T_1^{t,k} := P^k_{i^t_{\gamma(J_1^t)}} \cdots P^k_{i^t_2} P^k_{i^t_1}$, where $P^k_{i^t_l} = P_{C^k_{i^t_l}}$, for all $l = 1, 2, \cdots, \gamma(J_1^t)$ and every $t = 1, 2, \cdots, S_1$. We define the algorithmic mapping $\mathcal{P}_1^k := \sum_{t=1}^{S_1} \beta_t T_1^{t,k}$ with positive the weight vector β as in the previous section. We suppose also that every element of J_1 appears in at least one of the strings J_1^t. In the similar way, we get the algorithmic mapping $\mathcal{P}_2^k := \sum_{t=1}^{S_2} \eta_t T_2^{t,k}$ with the weight vector η as in the previous section. By Lemma 3.1, if $(I - \mathcal{P}_1^k)z = A^*(I - \mathcal{P}_2^k)Az = 0$ then we have only that $z \in \cap_{i=1}^N C_i^k$ and $Az \in \cap_{j=1}^M Q_j^k$. It is difficult to confirm that z is a solution of (1.1). So, we consider the following relaxed algorithmic scheme,

$$\begin{aligned} x^{k+1} &= \mathcal{P}_1^k(x^k - \gamma_k A^*(Ax^k - By^k)), \\ y^{k+1} &= \mathcal{P}_2^k(y^k + \gamma_k B^*(Ax^k - By^k)), \end{aligned} \tag{4.1}$$

where γ_k is chosen by (3.2) with the same conditions on ρ_k and ε_k.

The following Lemma is essential in proving convergence.

Lemma 4.1 [25] *Suppose h is a convex function on a Hilbert space H, then it is subdifferentiable everywhere and its subdifferentials are uniformly bounded subsets of H.*

Lemma 4.1 shows that the subdifferentials are bounded on bounded sets.

Theorem 4.1 *Let H_1, H_2, H_3, A, B and Γ be as in Theorem 3.1. Let C_i and Q_j, for each $i \in J_1$ and $j \in J_2$, be closed convex subsets in H_1 and H_2, that be defined by* (a1) *and* (a2). *Assume that there hold condition* (ε) *and* (3.2). *Then, the sequence $\{z^k\}$, defined by* (4.1), *converges weakly to a solution of* (1.1) *as $k \to \infty$.*

Proof. Take a point $z = [p, q] \in \Gamma$. Then, $Ap = Bq$. Since $C_i \subseteq C_i^k, Q_j \subseteq Q_j^k$, we have $p = P_i p = P_i^k p$, $q = P_j q = P_j^k q$ for all $i \in J_1, j \in J_2$ and $k \geq 1$. By the similar argument as in the proof of Theorem 3.1, we get inequality (3.6). Consequently, $\{z^k\}$, defined by (4.1) with $z^k = [x^k, y^k]$, is bounded with the limit (3.7). Moreover, we also get that

$$\lim_{k\to\infty} \|P_i^k x^k - x^k\| = 0 \ \forall i \in J_1 \quad \text{and} \quad \lim_{k\to\infty} \|P_j^k y^k - y^k\| = 0 \ \forall j \in J_2. \tag{4.3}$$

Next, from the definitions of C_i^k and Q_j^k, it follows that

$$\begin{aligned} c_i(x^k) &\leq \|\xi_i^k\| \|(I - P_i^k)x^k\|, \\ q_j(y^k) &\leq \|\theta_j^k\| \|(I - P_j^k)y^k\|. \end{aligned} \tag{4.4}$$

Since $\{z^k\}$ is bounded, $\{\xi_i^k\}$, $\{\theta_j^k\}$ are bounded and there exist subsequences $\{x^{k_l}\}$ of $\{x^k\}$ and $\{y^{k_l}\}$ of $\{y^k\}$ such that $\{x^{k_l}\}$ and $\{y^{k_l}\}$ converge weakly to a point $\tilde{x} \in H_1$ and a point $\tilde{y} \in H_2$, respectively, with $f(\tilde{x}, \tilde{y}) = 0$. Thus, from the bounded property of $\{\xi_i^k\}$ and $\{\theta_j^k\}$ together with (4.3) and (4.4) it follows that $c_i(\tilde{x}) \le 0$ and $q_j(\tilde{y}) \le 0$ for all $i \in J_1$ and $j \in J_2$. It means that $\tilde{x} \in \Gamma$. Analogously, we have that every weak cluster point of $\{z^k\}$ belongs to Γ, and hence, by Lemma 2.4, all the sequence $\{z^k\}$ converges weakly to a point in Γ. This completes the proof. □

5. A steepest-descent string-averaging scheme for the MSSEP

In order to obtain a strong convergence sequence from (3.1)-(3.2), we consider their combination with the steepest-descent method [27] for monotone mappings in Hilbert spaces, that was developed further in [26], [27] and [29] for the problem of common fixed points for a family of nonexpansive mappings.

Our scheme is defined by $z^1 = [x^1, y^1]$ with $x^1 \in H_1$ and $y^1 \in H_2$, any points, and

$$\begin{aligned} u^k &= \mathcal{P}_1(x^k - \gamma_k A^*(Ax^k - By^k)), \\ x^{k+1} &= (I - t_k F_1)u^k, \\ v^k &= \mathcal{P}_2(y^k + \gamma_k B^*(Ax^k - By^k)), \\ y^{k+1} &= (I - t_k F_2)v^k, \end{aligned} \tag{5.1}$$

where F_i is an η_i-strongly monotone and γ_i-strictly pseudocontractive mapping on H_i such that $\eta_i + \gamma_i > 1$ for $i = 1, 2$ and $\eta + \gamma > 1$ where $\eta = \min\{\eta_1, \eta_2\}$ and $\gamma = \max\{\gamma_1, \gamma_2\}$.

Clearly, method (5.1) can be re-written in the compact form,

$$z^{k+1} = (I - t_k F)\mathcal{P}(I - \gamma_k G^* G)z^k, \tag{5.2}$$

where $F = [F_1, F_2] : H \to H$ is η-strongly monotone and γ-strictly pseudocontractive and $\mathcal{P} = [\mathcal{P}_1, \mathcal{P}_2]$.

We have the following results.

Theorem 5.1 *Let H_1, H_2, H_3, A, B, C_i and Q_j with Γ be as in Theorem 2.1. Assume that there hold condition (ε) and (3.4). Then, the sequence $\{z^k\}$, defined by algorithmic scheme (5.2), as $k \to \infty$, converges strongly to the solution $z_* \in \Gamma$, satisfying the variational inequality problem*

$$\langle Fz_*, z_* - z\rangle \le 0 \ \forall z \in \Gamma.$$

Proof. First, we prove that $\{z^k\}$, defined by algorithmic scheme (5.2) is bounded. Put $s^k := \mathcal{P}(I - \gamma_k G^* G)z^k$. From the proof of Theorem 3.1, we get

$$\|s^k - z\|^2 \le \|z^k - z\|^2 - \rho_k(4 - \rho_k)\frac{f^2(x^k, y^k)}{a_k + \varepsilon_k} \le \|z^k - z\|^2. \tag{5.3}$$

Then, by virtue of Lemma 2.5 and (5.3),

$$\begin{aligned}\|z^{k+1}-z\| &= \|(I-t_kF)s^k-(I-t_kF)z-t_kFz\| \\ &\le (1-t_k\tau)\|s^k-z\|+t_k\|Fz\| \\ &\le (1-t_k\tau)\|z^k-z\|+t_k\|Fz\| \\ &\le \max\{\|z^1-z\|,\|Fz\|/\tau\},\end{aligned}$$

that means the boundedness of $\{z^k\}$, and

$$\begin{aligned}\|z^{k+1}-z\|^2 &= \|(I-t_kF)s^k-(I-t_kF)z-t_kFz\|^2 \\ &\le (1-t_k\tau)\|z^k-z\|^2-\rho\frac{f^2(x^k,y^k)}{a_k+\varepsilon_k}-2t_k\langle Fz,z^{k+1}-z\rangle.\end{aligned} \tag{5.4}$$

where ρ is some positive constant such that $(1-t_k\tau)\rho_k(2-\rho_k)\ge\rho$ for all $k\ge 1$. We need only to consider two cases.

Case 1 $\|z^{k+1}-z\|\le\|z^k-z\|$ for all $k\ge k_0$, large enough.

Then, there exists $\lim_{k\to\infty}\|z^k-z\|$. From (5.4) it follows that

$$0\le\rho\frac{f^2(x^k,y^k)}{a_k+\varepsilon_k}\le\|z^k-z\|^2-\|z^{k+1}-z\|^2+2t_k\|Fz\|\|z^{k+1}-z\|. \tag{5.5}$$

So, from (5.5), the existence of $\lim_{k\to\infty}\|z^k-z\|$ with the boundedness of $\{z^k\}$ and property of $\{\varepsilon_k\}$, it follows that $\lim_{k\to\infty}f(x^k,y^k)=0$. As in the proof of Theorem 3.1, we have that every weak cluster point of $\{z^k\}$ belongs Γ. Therefore,

$$\limsup_{k\to\infty}\langle Fz_*,z_*-z^k\rangle\le 0,\ \text{and hence,}\ \limsup_{k\to\infty}\langle Fz_*,z_*-z^{k+1}\rangle\le 0.$$

Now, from (5.4) we get that

$$\|z^{k+1}-z_*\|^2\le(1-t_k\tau)\|z^k-z_*\|^2+2t_k\langle Fz_*,z_*-z^{k+1}\rangle.$$

By Lemma 2.2, $\|z^k-z_*\|\to 0$ as $k\to\infty$.

Case 2. There exists a subsequence $\{k_l\}$ of $\{k\}$ such that $\|z^{k_l}-z\|<\|z^{k_l+1}-z\|$ for all $l\ge 0$.

Hence, by Lemma 2.3, there exists a nondecreasing sequence $\{m_k\}\subseteq\{k\}$ such that $m_k\to\infty$,

$$\|z^{m_k}-z\|\le\|z^{m_k+1}-z\| \quad \text{and} \quad \|z^k-z\|\le\|z^{m_k+1}-z\| \tag{5.6}$$

for each $k\ge 1$. Then, from (5.4) and the first inequality in (5.6), we know that

$$\|z^{m_k}-z\|^2\le\frac{2}{\tau}\langle Fz,z-z^{m_k+1}\rangle. \tag{5.7}$$

In this case, instead of (5.5), we get that

$$0 \le \rho \frac{f^2(x^{m_k}, y^{m_k})}{a_{m_k} + \varepsilon_{m_k}} \le \|z^{m_k} - z\|^2 - \|z^{m_k+1} - z\|^2 + 2t_{m_k}\|Fz\|\|z^{m_k+1} - z\|.$$

and hence, $\lim_{k\to\infty} f(x^{m_k}, y^{m_k}) = 0$. By the similar argument as in the proof for the case 1, any cluster point of $\{x^{m_k}\}$ belongs to Γ. Thus,

$$\limsup_{k\to\infty} \langle Fz_*, z_* - z^{m_k+1}\rangle \le 0,$$

which together with (5.7) implies that $\|z^{m_k} - z_*\| \to 0$ as $k \to \infty$. Now, from (5.4) with k and z replaced, respectively, by m_k and z_* it follows that $\|z^{m_k+1} - z_*\| \to 0$. Noting the second inequality in (5.6), $\|z^k - z_*\| \to 0$. The proof is completed. □

Remark We take $f = aI + (1-a)\overline{z}$ with a fixed $\overline{z} \in H$ and a fixed number $a \in (0,1)$. It is well known in [26] that $I - f$ is an η-strongly monotone and γ-strictly pseudocontractive mapping in H such that $\eta + \gamma > 1$. Replacing F in (5.2) by the mapping $I - f$, we obtain, instead of (1.9), the Halpern string-averaging scheme for the MSSEP,

$$z^{k+1} = t_k\overline{z} + (1 - t_k)\mathcal{P}(I - \gamma_k G^* G)z^k$$

with new $t_k := t_k(1-a)$, that converges strongly to a point in Γ without condition (t').

6. Numerical Examples

For computation, we consider the case $H_1 = \mathbb{E}^2$, $H_2 = \mathbb{E}^3$ and $H_3 = \mathbb{E}^4$; A and B are given bellow.

$$A = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.4 \\ 0.3 & 0.6 \\ 0 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.1 & 0.2 \\ 0 & 0.2 & 0.4 \\ 0 & 0.1 & 0 \end{bmatrix}.$$

We consider MSSEP (1.1) with $C_i = \{x \in \mathbb{E}^2 : \langle a^i, x\rangle \le \beta_i\}$, where $a^i = (1/i; -1)$ and $\beta_i = 0$, for $i = 1, \cdots, 10$, and $Q_j = \{y \in \mathbb{E}^3 : \|y - a^j\| \le 1\}$, where $a^j = (1/(j+1); 1/(j+1); 1/(j+1))$ for $j = 1, \cdots, 15$. Clearly, problem (1.1) with the data above has many solutions. So, in order to verify the convergence to a solution, that we do not know, for algorithmic scheme (3.1)–(3.2), we use the errors: $error1 := \|x^{k+1} - x^k\|/\|x^k\|$ and $error2 := \|y^{k+1} - y^k\|/\|y^k\|$ with $\rho_k = 3 + 1/(k+1)$, $\varepsilon_k = 1$ for all $k \ge 1$, $x^1 = (-3.0; 3.0)$ and $y^1 = (-2.0; -2.5; 2.0)$. Put $\tilde{\mathcal{P}}_1 = (P_{C_5}\cdots P_{C_1} + P_{C_{10}}\cdots P_{C_6})/2$ and $\tilde{\mathcal{P}}_2 = (P_{Q_5}\cdots P_{Q_1} + P_{Q_{10}}\cdots P_{Q_6} + P_{Q_{15}}\cdots P_{Q_{11}})/3$. The numerical results with different $\mathcal{P}_1$ and $\mathcal{P}_2$ are given in the following tables.

k	$error1$	$error2$	k	$error1$	$error2$
10	0.0012953412	0.0084375860	100	0.0000584719	0.0004042637
20	0.0005700299	0.0049270390	200	0.0000189949	0.0001356754
30	0.0003496738	0.0030891459	300	0.0000100827	0.0000746127
40	0.0002398504	0.0020088602	400	0.0000064987	0.0000495669
50	0.0001747594	0.0013715507	500	0.0000046404	0.0000363808

Table 1. Method (3.1)–(3.2) with $\mathcal{P}_1 = \sum_{i=1}^{10} P_{C_i}/10$ and $\mathcal{P}_2 = \sum_{j=1}^{15} P_{Q_j}/15$

k	$error1$	$error2$	k	$error1$	$error2$
10	0.0009321189	0.0054130662	100	0.0000422591	0.0002421531
20	0.0003776241	0.0021946777	200	0.0000164338	0.0000934767
30	0.0002192796	0.0012719729	300	0.0000095113	0.0000504397
40	0.0001483827	0.000858825440	400	0.0000064435	0.0000367375
50	0.0001093893	0.000631803850	500	0.0000047357	0.0000272121

Table 2. Method (3.1)–(3.2) with $\mathcal{P}_1 = P_{C_{10}} \cdots P_{C_1}$ and $\mathcal{P}_2 = P_{Q_{15}} \cdots P_{Q_1}$

k	$error1$	$error2$	k	$error1$	$error2$
10	0.0008866128	0.0049620475	100	0.0000421089	0.0002325655
20	0.0003674546	0.0020570113	200	0.0000164619	0.0000903616
30	0.0002152168	0.0012024210	300	0.0000095401	0.0000523899
40	0.0001463201	0.0008157362	400	0.0000064555	0.0000356467
50	0.0001081992	0.0006019969	500	0.0000047286	0.0000263883

Table 3. Method (3.1)–(3.2) with $\mathcal{P}_1 = \tilde{\mathcal{P}}_1$ and $\mathcal{P}_2 = \tilde{\mathcal{P}}_2$

k	$error1$	$error2$	k	$error1$	$error2$
10	0.0577563243	0.0067677696	100	0.0070196412	0.0008262614
20	0.0234714242	0.0041274345	200	0.0038312744	0.0003778241
30	0.0181178430	0.0028980667	300	0.0027201046	0.0002397887
40	0.0147753768	0.0021985851	400	0.0021569324	0.0001743768
50	0.0124738808	0.0017529726	500	0.0018126310	0.0001363898

Table 4. Method (3.1)–(3.2) with $\mathcal{P}_1 = \sum_{i=1}^{10} P_{C_i}/10$ and $\mathcal{P}_2 = \tilde{\mathcal{P}}_2$

Now, for testing algorithmic scheme (4.1) we take $H_1 = \mathbb{E}^2$, $H_2 = H_3 = \mathbb{E}^3$ and A and B are defined as the above with deleting fourth rows. The sets C_i and Q_j are given by

$$
\begin{aligned}
C_1 &= \{x \in \mathbb{E}^2 : x_1^2/2 + x_2 \le 0\},\\
C_2 &= \{x \in \mathbb{E}^2 : x_1 + x_2^2/2 - 1 \le 0\},\\
C_3 &= \{x \in \mathbb{E}^2 : x_1 + x_2 - 3 \le 0\},\\
C_4 &= \{x \in \mathbb{E}^2 : x_1^2/2 + x_2^2/2 - 4 \le 0\}
\end{aligned}
$$

and

$$Q_1 = \{y \in \mathbb{E}^3 : y_1^2/2 + y_2 + y_3 - 1 \le 0\},$$
$$Q_2 = \{y \in \mathbb{E}^3 : y_1 + y_2^2/2 + y3 - 2 \le 0\},$$
$$Q_3 = \{y \in \mathbb{E}^3 : y_1^2/2 + y_2^2/2 + y_3^2/2 - 3 \le 0\}.$$

By using algorithmic scheme (4.1) with the same data as the above and a new value $\rho_k = 0.4 + 1/(k+2)$, we obtain the following numerical tables, Tables 5 and 6.

k	$error1$	$error2$	k	$error1$	$error2$
10	0.0355614672	0.0119739663	100	0.0002863193	0.0007033014
20	0.0090352165	0.0056429197	200	0.0001688489	0.0001306101
30	0.0032744843	0.0039105849	300	0.0000888721	0.0000497822
40	0.0013332436	0.0029413076	400	0.0000560462	0.0000275918
50	0.0005198815	0.0022639672	500	0.0000398194	0.0000181807

Table 5. Method (4.1) with $\mathcal{P}_1^k = \sum_{i=1}^4 P_i^k/4$ and $\mathcal{P}_2^k = \sum_{j=1}^3 P_j^k/3$

k	$error1$	$error2$	k	$error1$	$error2$
10	0.0311695625	0.0152272465	100	0.0000067994	0.0003112079
20	0.0064277037	0.0069224575	200	0.0000059154	0.0000846789
30	0.0019876049	0.0036999871	300	0.0000021392	0.0000427758
40	0.0007273143	0.0021601957	400	0.0000005677	0.0000270271
50	0.0002890575	0.0013644448	500	0.0000001846	0.0000191596

Table 6. Method (4.1) with $\mathcal{P}_1^k = P_4^k \cdots P_1^k$ and $\mathcal{P}_2^k = P_3^k \cdots P_1^k$

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