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HAMILTONIAN DECOMPOSITIONS OF COMPLETE 4-PARTITE 3-UNIFORM HYPERGRAPHS

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Abstract

Based on the definition of Hamiltonian cycles given by Katona and Kierstead, we provide a construction of Hamiltonian decompositions of the complete 4-partite 3-uniform hypergraph $K^{(3)}_{4(2m)}$, where $2m$ is the size of each partite set.

1 Introduction

A *Hamiltonian decomposition* of a hypergraph is a partition of its hyperedge set into mutually disjoint Hamiltonian cycles. The definition of a Hamiltonian cycle can be extended to hypergraphs in various ways. The definition in this paper based on a Hamitonicity of cycles for k-uniform hypergraphs $\mathcal{H}(V, E)$

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of order n given by Katona and Kierstead [5] : a *Hamiltonian cycle* of $\mathcal{H}$ is a cyclic ordering $C = (v_1\ v_2\ \ \dots\ v_n)$ of all n elements of V such that k consecutive vertices form a hyperedge in E.

The existence of problem of Hamiltonian decompositions have been studied widely for complete 3-uniform hypergraphs, $K_n^{(k)}$, for example, $K_9^{(4)}$ and $K_n^{(3)}$ for some admissible $n \leq 46,\ n = 2^m$ and $m \geq 2$ (see [1, 6, 4, 8]). Additionally, several authors studied the problem for the complete multipartite k-uniform hypergraphs, defined as follows :

Definition 1.1. A *complete multipartite k-uniform hypergraph* $K^{(k)}_{n_1,n_2,\dots,n_t}$ or $K^{(k)}_{n_1,n_2,\dots,n_t}(V_1, V_2, \dots,\ V_t)$ is a hypergraph with vertex set $V = V_1 \cup V_2 \cup \cdots \cup V_t$ where $|V_i| = n_i$ for all $i \in \{1, 2, \dots, t\}$, and

$$E(\mathcal{H}) = \{e : e \subseteq V, |e| = k \text{ and } |e \cap V_i| < k \text{ for } i \in \{1, 2, \dots, t\}\}.$$

In particular, if $n_i = n$ for all $i \in \{1, 2, \dots, t\}$, then $K^{(k)}_{\underbrace{n, n, \dots n}_{t}}$ is denoted by $K^{(k)}_{t(n)}$.

In literature, the problem of Hamiltonian decompositions of $K^{(k)}_{t(n)}$ has been investigated only for the case $k = 3$ and $t = 2$ and 3. Wang and Jirimutu [7] studied on $K^{(3)}_{n,n}$ when n is a prime number in 2001. Later on, Xu and Wang [8] provided a complete study for all $n \geq 2$ in 2002. The study for complete tripartite 3-uniform hypergraphs, $K^{(3)}_{n,n,n}$, was also completed by Boonklurb *et al.* [2] in 2015. Continuing along this line, we are interested in the case $t = 4$ and n is any even positive integer. In other words, we provide a construction of Hamiltonian decompositions of $K^{(3)}_{4(2m)}$ for all positive integer m.

2 Hamiltonian decompositions of $K^{(3)}_{4(2m)}$

We first classify hyperedges of $K^{(3)}_{4(2m)}$ into two types. Let e be a hyperedge of $K^{(3)}_{4(2m)}$, if e contains at most one vertex from each partite set, e is then called a hyperedge of *Type* 1, otherwise (that is e contains two vertices from a partite set) e is called a hyperedge of *Type* 2. The following notations will be used for the rest of the paper unless state otherwise.

Notations

$2m$ is the size of each partite set,

$[n]$ is the set of integers $\{1, 2, \ldots, n\}$,

$\mathcal{T}_i(K^{(3)}_{4(2m)})$ is the subhypergraph of $K^{(3)}_{4(2m)}$ which consists of all hyperedges of Type i for $i = 1, 2$,

the vertex set of $K^{(3)}_{4(2m)}$ is $V_1 \cup V_2 \cup V_3 \cup V_4$ where $V_i = \{a^i_1, a^i_2, \ldots, a^i_{2m}\}$ for $i \in \{1, 2, 3, 4\}$,

$E(\mathcal{H})$ is the hyperedge set of hypergraph $\mathcal{H}$.

We will present the construction by decomposing $\mathcal{T}_1(K^{(3)}_{4(2m)})$ and $\mathcal{T}_2(K^{(3)}_{4(2m)})$ into Hamiltonian cycles in Sections 2.1 and 2.2 separately.

Recall that a Hamiltonian cycle C of $K^{(3)}_{4(2m)}$ is a cycle in which any three consecutive vertices form a hyperedge. In our construction, we write a cycle C as $(P_1\ P_2\ \ldots\ P_s)$ if vertices along the cycle C are partitioned into *paths* P_j (a sequence of vertices) along this cycle.

On top of that, each hyperedge in C is called an *inline hyperedge* if it is a hyperedge within a path or, a *joint hyperedge* if it contains vertices from some two consecutive paths.

One of the main tools for our construction is a 1*-factorization* of a graph which is a partition of a graph into 1-factors (1-regular spanning subgraphs). Although, graphs are 2-uniform hypergraphs, we use the usual notations of graphs such as K_n and $K_{n,n}$ for $K^{(2)}_n$ and $K^{(2)}_{n,n}$, respectively. Also, we denote a complete graph K_n on the vertex set V by $K_n(V)$. The following are classic results published first time in [3].

Theorem 2.1. [3]

(*i*) *The complete graph K_n has a* 1*-factorization whenever n is even,*

(*ii*) *The complete bipartite graph $K_{n,n}$ has a* 1*-factorization for all positive integer n.*

In this paper, we refer to a 1-factor of a graph as its edge set. In particular, if a 1-factor F of $K_{2m}([2m])$ is written as $\{\{j, f(j)\} : j \in \{1, 2, \ldots, m\}\}$, then the vertex set $[2m]$ is relabeled to be $\{1, 2, \ldots, 1, f(1),\ f(2), \ldots, f(m)\}$.

2.1 Hamiltonian decompositions of $\mathcal{T}_1(K^{(3)}_{4(2m)})$

This section considers all hyperedges of Type 1. We establish a stronger result by giving a Hamiltonian decomposition of the subhypergraph $\mathcal{T}_1(K^{(3)}_{4(n)})$ for all positive integers n, instead of even ones. The construction uses a 1-factorization of $K_{n,n}([n],[n])$. In Theorem 2.2, the construction creates each Hamiltonian cycle consisting of n paths of order four; each path contains exactly one vertex from each partite set.

Theorem 2.2. *The hypergraph $\mathcal{T}_1(K^{(3)}_{4(n)})$ has a Hamiltonian decomposition for all positive integer n.*

Proof. Let $\mathscr{F}$ be a 1-factorization of the complete bipartite graph $K_{n,n}([n],[n])$ which exists by Theorem 2.1. In this proof, we will construct a collection

$$\mathscr{C} = \{C_t(F) : t \in \{0,1,\ldots,n-1\}, F \in \mathscr{F}\}$$

which will be later showed that it is a Hamiltonian decomposition of $\mathcal{T}_1(K^{(3)}_{4(n)})$.

Let $F = \{\{i, f(i)\} : i \in \{1,2,\ldots,n\}\}$ be any 1-factor in $\mathscr{F}$. There are a total of n Hamiltonian cycles in $\mathscr{C}$ constructed from F, where each cycle is composed of n paths of order four as follows. For $t \in \{0,1,\ldots,n-1\}$, the cycle $C_t(F) = (P_1^t\ P_2^t\ \cdots\ P_n^t)$ such that for $j \in \{1,2,\ldots,n\}$,

$$P_j^t = a^1_{j+t}\ \ a^2_{f(j+t)}\ \ a^3_j\ \ a^4_{f(j)}$$

where $j+i$ is considered in the modulus n. Thus,

$$\begin{array}{rcccc} C_t(F) = (& a^1_{1+t} & a^2_{f(1+t)} & a^3_1 & a^4_{f(1)} \\ & a^1_{2+t} & a^2_{f(2+t)} & a^3_2 & a^4_{f(2)} \\ & \vdots & \vdots & \vdots & \vdots \\ & a^1_{n-1+t} & a^2_{f(n-1+t)} & a^3_{n-1} & a^4_{f(n-1)} \\ & a^1_{n+t} & a^2_{f(n+t)} & a^3_n & a^4_{f(n)} \quad). \end{array}$$

Since F is a 1-factor, all $4n$ vertices in $C_t(F)$ are distinct. This yields that $C_t(F)$ is a Hamiltonian cycle of $K^{(3)}_{4(n)}$ for all t. Besides, three consecutive vertices in $C_t(F)$ always come from three different partite sets; so, $C_t(F)$ is a Hamiltonian cycle of $\mathcal{T}_1(K^{(3)}_{4(n)})$.

Now, we will claim that the collection $\mathscr{C}$ is a decomposition of $\mathcal{T}_1(K_{4(n)}^{(3)})$. Denote the set of all hyperedges of all cycles in $\mathscr{C}$ by $E(\mathscr{C})$. Hence $E(\mathscr{C})$ contains a total of $4n^3$ hyperedges of Type 1 counted repeatly. Since the number of hyperedges of Type 1 in $K_{t(n)}^{(3)}$ is also $4n^3$, it suffices to show that each hyperedge of Type 1 in $K_{t(n)}^{(3)}$ is covered at least once in $E(\mathscr{C})$.

Since there are only four possible ways to choose three partite sets from the four partite sets of $K_{4(n)}^{(3)}$, we can classify hyperedges of Type 1 into four groups depending on partite sets containing their vertices. First, consider hyperedges of Type 1 containing vertices from V_1, V_2 and V_3. Let e be such a hyperedge written $e = \{a_u^1, a_v^2, a_w^3\}$ where $u, v, w \in [n]$. We claim that e is covered at least once in $E(\mathscr{C})$. Now, consider u and v as vertices from two distinct partite sets of $K_{n,n}([n],[n])$. Since $\mathscr{F}$ is a 1-factorization of $K_{n,n}([n],[n])$, there exists unique 1-factor $F' = \{\{j, f'(j)\} : j \in \{1,2,\ldots,m\}\} \in \mathscr{F}$ such that $\{u, v\} = \{i, f'(i)\}$ for some i. Then $e = \{a_i^1, a_{f'(i)}^2, a_w^3\}$. In $C_t(F') = (\overline{P}_1^t\ \overline{P}_2^t\ \cdots\ \overline{P}_n^t)$, the first inline hyperedge of each path $\overline{P}_j^t$ for $j \in \{1,2,\ldots,m\}$ is $\{a_{j+t}^1,\ a_{f'(j+t)}^2,\ a_j^3\}$. Then $\bigcup_{t=1}^n E(C_t(F'))$ contains $\{\{a_k^1, a_{f'(k)}^2, a_\ell^3\} \colon k, \ell \in \{1,2,\ldots,n\}\}$. Hence $e \in \bigcup_{t=1}^n E(C_t(F')) \subseteq E(\mathscr{C})$ as claimed.

If e is a hyperedge of Type 1 with the partite sets $\{V_2, V_3, V_4\}$, $\{V_3, V_4, V_1\}$ or $\{V_4, V_1, V_2\}$, then we can prove in a similar fashion that e is covered by at least one cycle in $E(\mathscr{C})$, say $C_t(F) = (P_1^t\ P_2^t\ \cdots\ P_n^t)$, where $F = \{\{i, f(i)\} : i \in \{1,2,\ldots,n\}\}$; it is provided by the fact that each path P_j^t also contains an inline hyperedge $\{a_{f(j+t)}^2,\ a_j^3,\ \ a_{f(j)}^4\}$, and there are exactly two joint hyperedges connecting P_j^t and P_{j+1}^t, $\{a_j^3,\ a_{f(j)}^4,\ a_{j+1+t}^1\}$ and $\{a_{f(j)}^4,\ a_{j+1+t}^1,\ a_{f(j+1+t)}^2\}$.

Therefore $\mathscr{C}$ is a Hamiltonian decomposition of $\mathcal{T}_1(K_{4(n)}^{(3)})$. □

2.2 Hamiltonian decompositions of $\mathcal{T}_2(K_{4(2m)}^{(3)})$

In this section, we decompose the subhypergraph $\mathcal{T}_2(K_{4(2m)}^{(3)})$ containing all hyperedges of Type 2. The construction uses the following tools:

the collection of 4-tuples $\mathscr{D} = \{(1,2,3,4), (1,3,4,2), (1,4,2,3)\}$ and

a 1-factorization $\mathscr{F}$ of $K_{2m}([2m])$ which always exists by Theorem 2.1.

Now, we aim to establish the following two collections of cycles in $K^{(3)}_{4(2m)}$ which are depending on the parity of m,
$\mathscr{C} = \{C_t(D,F) : t \in \{0,1,\ldots,m-1\}, D \in \mathscr{D},$ and $F \in \mathscr{F}\}$ for odd m, and
$\overline{\mathscr{C}} = \{C_t(D,F), \overline{C}_t(D,F) : t \in \{0,1,\ldots,\frac{m}{2}-1\}, D \in \mathscr{D},$ and $F \in \mathscr{F}\}$ for even m.

Thus, each collection will contain $3m(2m-1)$ cycles. For the construction, let D be any tuple in $\mathscr{D}$, and F any 1-factor of $K_{2m}([2m])$ in $\mathscr{F}$, written

$$D = (p,q,r,s) \text{ and } F = \{\{j, f(j)\} : j \in \{1,2,\ldots,m\}\},$$

consequently, the vertex set of K_{2m} is relabel to be $\{1,2,\ldots,1,f(1),f(2),\ldots,f(m)\}$. We will construct m Hamiltonian cycles of $\mathcal{T}_2(K^{(3)}_{4(2m)})$ in $\mathscr{C}$ from D and F when m is odd in Section 2.2.1, namely

$$C_0(D,F),\ C_1(D,F),\ \ldots,\ C_{m-1}(D,F), \text{ and}$$

m Hamiltonian cycles of $\mathcal{T}_2(K^{(3)}_{4(2m)})$ in $\overline{\mathscr{C}}$ from D and F when m is even in Section 2.2.2, namely

$$C_0(D,F),\ C_1(D,F),\ \ldots,\ C_{\frac{m}{2}-1}(D,F),\ \overline{C}_0(D,F),\ \overline{C}_1(D,F),\ \ldots,\ \overline{C}_{\frac{m}{2}-1}(D,F).$$

Then we later show that both collections are Hamiltonian decompositions of $\mathcal{T}_2(K^{(3)}_{4(2m)})$.

2.2.1 m is odd.

Let m be an odd integer. We define $C_t(D,F)$ where $t \in \{0,1,\ldots,m-1\}$ to consist of two paths of order $4m$, written

$$C_t(D,F) = (P_1^t \ \ P_2^t)$$

such that for $j = 1,2$,

$$P_j^t = \begin{array}{cccc}
a^x_{1+t} & a^x_{f(1+t)} & a^y_{f(m+t)} & a^y_{m+t} \\
a^x_{2+t} & a^x_{f(2+t)} & a^y_{f(m-1+t)} & a^y_{m-1+t} \\
\vdots & \vdots & \vdots & \vdots \\
a^x_{\frac{m+1}{2}+t} & a^x_{f(\frac{m+1}{2}+t)} & a^y_{f(\frac{m+1}{2}+t)} & a^y_{\frac{m+1}{2}+t} \\
\vdots & \vdots & \vdots & \vdots \\
a^x_{m-1+t} & a^x_{f(m-1+t)} & a^y_{f(2+t)} & a^y_{2+t} \\
a^x_{m+t} & a^x_{f(m+t)} & a^y_{f(1+t)} & a^y_{1+t},
\end{array}$$

where

$$(x,y) = \begin{cases} (p,q), & \text{if } j = 1, \\ (r,s), & \text{if } j = 2. \end{cases}$$

We say that $C_t(D,F)$ is the t^{th} *rotation* of $C_0(D,F)$. In other words, $C_0(D,F)$ is an *initial cycle* which is rotated $m-1$ times to create additional $m-1$ cycles.

Example 2.3. *An illustration of $C_0(D,F)$ which are in the construction of $K^{(3)}_{4(2m)}$ when $m=5$, $D=(1,3,4,2)$ and $F=\{\{j,f(j)\} : j \in \{1,2,3,4,5\}\}$.* In Figure 1, each vertex a^x_ℓ in the cycle $C_0(D,F)$ is represented by its subscript ℓ. The solid lines indicate two consecutive vertices in the same path, while the dash lines indicate two consecutive vertices from different paths.

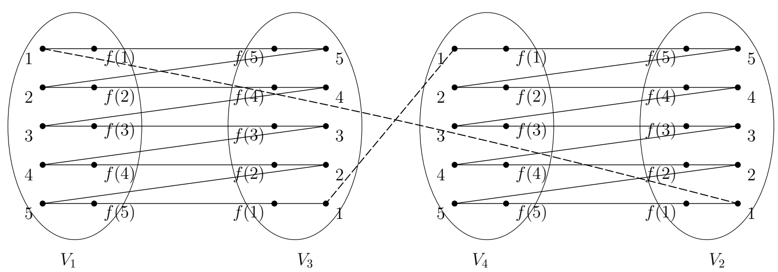

Figure 1: $C_0(D,F)$ of $\mathcal{T}_2(K^{(3)}_{4(10)})$.

Lemma 2.4. *Let $D \in \mathscr{D}$, $F \in \mathscr{F}$ and $t \in \{0,1,\ldots,m-1\}$. $C_t(D,F)$ is a Hamiltonian cycle of $\mathcal{T}_2(K^{(3)}_{4(2m)})$.*

Proof. Write $D=(p,q,r,s) \in \mathscr{D}$, we have that P^t_1 consists of $4m$ vertices from V_p and V_q and, P^t_2 consists of $4m$ vertices from V_r and V_s. Since (p,q,r,s) is a permutation of $\{1,2,3,4\}$ and F is a 1-factor of K_{2m}, the $8m$ vertices in $C_t(D,F)$ are all distinct. Furthermore, the construction yields that any three consecutive vertices in $C_t(D,F)$ are always from only two partite sets. Therefore all hyperedges in $C_t(D,F)$ are of Type 2. $\square$

Next, let us observe a certain property of hyperedges in $\mathcal{T}_2(K^{(3)}_{4(2m)})$. Let e be a hyperedge in $\mathcal{T}_2(K^{(3)}_{4(2m)})$, say e contains two vertices from V_x and one vertex from V_y where $x \neq y$, written $e=\{a^x_u, a^x_v, a^y_w\}$. Now consider u,v as vertices in $K_{2m}([2m])$. Since $\mathscr{F}$ is a 1-factorization of $K_{2m}([2m])$, there exists unique $F = \{\{j, f(j)\} : j \in \{1, 2, \ldots, m\}\} \in \mathscr{F}$ such that $\{u, v\} = \{i, f(i)\}$ for unique i. In such vertex set relabeled by F, we also consider w as another vertex, then there exists unique j such that $w = j$ or $f(j)$. Thus e must be one of the followings :

$$\{a_i^x, a_{f(i)}^x, a_j^y\} \text{ or } \{a_i^x, a_{f(i)}^x, a_{f(j)}^y\}.$$

Consequently, given two partite sets in order, we can define the *length* of each hyperedge of Type 2 from such partite sets as follows.

Definition 2.5. Let $(x, y) \in \{(p, q) \colon p, q \in \{1, 2, 3, 4\}, p \neq q\}$ and e a hyperedge of Type 2 with two partite sets V_x and V_y. Then there exist unique $F = \{\{j, f(j)\} : j \in \{1, 2, \ldots, m\}\} \in \mathscr{F}$ and unique $i, j \in \{1, 2, \ldots, m\}$ such that e can be written in one of the following four distinct forms,

$$\{a_i^x, a_{f(i)}^x, a_j^y\}, \quad \{a_i^x, a_{f(i)}^x, a_{f(j)}^y\}, \quad \{a_j^y, a_{f(j)}^y, a_i^x\}, \text{ and } \{a_j^y, a_{f(j)}^y, a_{f(i)}^x\}.$$

Define the *length with respect to* (x, y) of hyperedge e by

$$\mathcal{L}_{(x,y)}(e) = \begin{cases} i - j, & \text{if } e = \{a_i^x, a_{f(i)}^x, a_j^y\} \text{ or } \{a_j^y, a_{f(j)}^y, a_i^x\}, \\ (i - j)', & \text{if } e = \{a_i^x, a_{f(i)}^x, a_{f(j)}^y\} \text{ or } \{a_j^y, a_{f(j)}^y, a_{f(i)}^x\}. \end{cases}$$

where $i - j$ and $(i - j)'$ are considered in the modulus m. Then there are $2m$ possible lengths in $\{0, 1, \ldots,\ m - 1, 0', 1', \ldots, (m - 1)'\}$ denoted by $\mathscr{L}$.

Moreover, in the construction, as the partite sets of vertices are determined by $D \in \mathscr{D}$, we consider the length of hyperedges in $C_t(D, F)$ according to D as follows.

Definition 2.6. Let $D = (p, q, r, s) \in \mathscr{D}, F \in \mathscr{F}$ and $e \in C_t(D, F)$. Then e is a hyperedge of Type 2 with V_x and V_y for some $(x, y) \in \{(p, q), (q, r), (r, s), (s, p)\}$. The *length of a hyperedge* e is $\mathcal{L}_{(x,y)}(e)$.

Example 2.7. Figure 2 illustrates the lengths of hyperedges in the cycle $C_0(D, F)$ in Example 2.3. As each three consecutive vertices along the cycle form a hyperedge, we label its length at the middle vertex of such hyperedge.

Definition 2.8. *A hyperedge with* $\langle p, q \rangle$*-partite sets* is a hyperedge of Type 2 containing two vertices in V_p and one vertex in V_q.

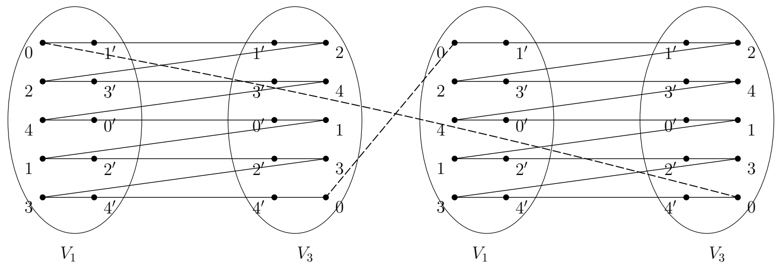

Figure 2: Lengths of hyperedges in $C_0(D,F)$ of $\mathcal{T}_2(K^{(3)}_{4(10)})$.

The next lemma discusses the lengths of hyperedges in $C_0(D,F)$, which yields the same result for other cycles in $\mathscr{C}$ as a rotation of an initial cycle preserves the lengths of hyperedges in a new cycle.

Lemma 2.9. *Let $D=(p,q,r,s)\in\mathscr{D}, F\in\mathscr{F}$, $I_D=\{(p,q),(r,s)\}$ and $J_D=\{(q,r),(s,p)\}$. The cycle $C_0(D,F)$ consists of the following:*

(i) *for $(x,y)\in I_D$, one inline hyperedge with $\langle x,y\rangle$-partite sets of length λ, and one inline hyperedge with $\langle y,x\rangle$-partite sets of length λ, for each $\lambda\in\mathscr{L}\setminus\{0\}$,*

(ii) *for $(x,y)\in J_D$, one joint hyperedge with $\langle x,y\rangle$-partite sets of length 0, and one joint hyperedge with $\langle y,x\rangle$-partite sets of length 0.*

Proof. Let $F\in\mathscr{F}$, written $F=\{\{j,f(j)\}:j\in\{1,2,\dots,m\}\}$. Let $e_1,e_2,\dots,e_{8m}$ be $8m$ hyperedges around the cycle $C_0(D,F)$ orderly, beginning with the first four inline hyperedges $e_1=\{a^p_1,\ a^p_{f(1)},a^q_{f(m)}\}, e_2=\{a^p_{f(1)},a^q_{f(m)},a^q_m\}, e_3=\{a^q_{f(m)},a^q_m,a^p_2\}$, $e_4=\{a^q_m,a^p_2,a^q_{f(2)}\}$ and so on. Note that e_{4m-1},e_{4m},e_{8m-1} and e_{8m} are joint hyperedges while the others $8m-4$ hyperedges are inline hyperedges.

By our construction, the lengths of inline hyperedges of P^0_1 and P^0_2 have the same spectrum. In particular, for $\ell\in\{1,2,\dots,4m-2\}$,

$$\mathcal{L}_{(p,q)}(e_\ell)=\mathcal{L}_{(r,s)}(e_{4m+\ell}).$$

For $\ell\in\{4m-1,4m\}$, e_ℓ and $e_{4m+\ell}$ are joint hyperedges satisfying $\mathcal{L}_{(q,r)}(e_\ell)=\mathcal{L}_{(s,p)}(e_{4m+\ell})$. Then it suffices to determine the lengths the first $4m$ hyperedges.

It is clear that $\mathcal{L}_{(q,r)}(e_{4m-1}) = 0$ and $\mathcal{L}_{(q,r)}(e_{4m}) = 0$, that is, the lengths of joint hyperedges are all 0. Thus (ii) is proved.

For other hyperedges, Table 1 reveals the length of inline hyperedge e_ℓ where $\ell \in \{1, 2, \ldots, 4m-2\}$.

e_{4d+k} where $d \in \{0,1,\ldots,m-1\}$ and $4d+k \leq 4m-2$				
k	e_{4d+k}			$\mathcal{L}_{(p,q)}(e_{4d+k})$
1	$\{a^p_{1+d},$	$a^p_{f(1+d)},$	$a^q_{f(m-d)}\}$	$(1+2d \pmod{m})'$
2	$\{a^p_{f(1+d)},$	$a^q_{f(m-d)},$	$a^q_{m-d}\}$	$(1+2d \pmod{m})'$
3	$\{a^q_{f(m-d)},$	$a^q_{m-d},$	$a^p_{2+d}\}$	$2+2d \pmod{m}$
4	$\{a^q_{m-d},$	$a^p_{2+d},$	$a^p_{f(2+d)}\}$	$2+2d \pmod{m}$

Table 1: Lengths of $e_1, e_2, \ldots, e_{4m-2}$.

With some abuse of notation, we refer to $(\lambda+2)'$ as $\lambda'+2$. Then it can be noticed further that the sequence of the lengths of inline hyperedges satisfies a recurrence relation

$$\mathcal{L}_{(p,q)}(e_\ell) = \mathcal{L}_{(p,q)}(e_{\ell-4}) + 2$$

for $\ell \in \{5, 6, \ldots, 4m-2\}$ where $\mathcal{L}_{(p,q)}(e_1) = 1'$, $\mathcal{L}_{(p,q)}(e_2) = 1'$, $\mathcal{L}_{(p,q)}(e_3) = 2$, $\mathcal{L}_{(p,q)}(e_4) = 2$.

Now all inline hyperedges with $\langle p,q\rangle$-partite sets in $C_0(D,F)$ are hyperedges e_ℓ for all $\ell \equiv 0, 1 \pmod 4$ and $\ell \leq 4m-1$. Since the modulus m is odd, the recurrence relation yields that the lengths of such $2m-1$ inline hyperedges span the set $\mathscr{L} \setminus \{0\}$ (see Tables 2 and 3). That is,

$$\{\mathcal{L}_{(p,q)}(e_\ell) \colon \ell \equiv 0, 1 \pmod 4, \ell \in \{0, 1, \ldots, 4m-2\}\} = \mathscr{L} \setminus \{0\}.$$

ℓ	1	5	9	$\cdots$	$\frac{m-3}{2}$	$\frac{m+1}{2}$	$\frac{m+9}{2}$	$\cdots$	$4m-7$	$4m-3$
$\mathcal{L}_{(p,q)}(e_\ell)$	$1'$	$3'$	$5'$	$\cdots$	$(m-2)'$	$0'$	$2'$	$\cdots$	$(m-3)'$	$(m-1)'$

Table 2: The lengths of m inline hyperedges in $\{e_\ell \colon \ell \equiv 1 \pmod 4, \ell \in \{0, 1, \ldots, 4m-2\}\}$.

Similarly, inline hyperedges with $\langle q,p\rangle$-partite sets in $C_0(D,F)$ are hyperedges e_ℓ for all $\ell \equiv 2, 3 \pmod 4$ and $\ell \leq 4m-2$ which also have lengths

ℓ	4	8	12	$\cdots$	$\frac{m-1}{2}$	$\frac{m+7}{2}$	$\frac{m+15}{2}$	$\cdots$	$4m-8$	$4m-4$
$\mathcal{L}_{(p,q)}(e_\ell)$	2	4	6	$\cdots$	$m-1$	1	3	$\cdots$	$m-4$	$m-2$

Table 3: The lengths of $m-1$ inline hyperedges in $\{e_\ell \colon \ell \equiv 0 \pmod 4, \ell \in \{0,1,\dots,4m-2\}\}$.

spanning the set $\mathscr{L}\setminus\{0\}$ as follows.

$$\{\mathcal{L}_{(p,q)}(e_\ell) \colon \ell \equiv 2,3 \pmod 4, \ell \in \{0,1,\dots,4m-2\}\} = \mathscr{L}\setminus\{0\}.$$

Hence for $\lambda \in \mathscr{L}\setminus\{0\}$, $C_0(D,F)$ contains exactly one hyperedge with $\langle p,q\rangle$-partite sets of length λ, and one hyperedge with $\langle q,p\rangle$-partite sets of length λ. Therefore (i) is proved. □

Theorem 2.10. *The subhypergraph $\mathcal{T}_2(K^{(3)}_{4(2m)})$ has a Hamiltonian decomposition when m is odd.*

Proof. Let $\mathscr{D} = \{(1,2,3,4),(1,3,4,2),(1,4,2,3)\}$, $\mathscr{F}$ a 1-factorization of K_{2m} and

$$\mathscr{C} = \{C_t(D,F) : t \in \{0,1,\dots,m-1\}, D \in \mathscr{D} \text{ and } F \in \mathscr{F}\}.$$

By Lemma 2.4, $\mathscr{C}$ is a collection of Hamiltonian cycles of $\mathcal{T}_2(K^{(3)}_{4(2m)})$. It remains to show that $\mathscr{C}$ is a decomposition of $\mathcal{T}_2(K^{(3)}_{4(2m)})$.

First, we consider an essential property of $\mathscr{D}$. The following two collections I and J contain ordered pairs induced by $\mathscr{D}$;

$$I = \{(p,q),(r,s) \colon (p,q,r,s) \in \mathscr{D}\} = \{(1,2),(1,3),(1,4),(3,4),(4,2),(2,3)\}, \text{ and}$$

$$J = \{(q,r),(s,p) \colon (p,q,r,s) \in \mathscr{D}\} = \{(2,3),(3,4),(4,2),(4,1),(2,1),(3,1)\}.$$

For $D \in \mathscr{D}$, $E(\{D\},\mathscr{F})$ stands for the collection of hyperedges of all cycles constructed by D and $\mathscr{F}$. Given $D = (p,q,r,s)$, $E(\{D\},\mathscr{F})$ contains

inline hyperedges with partite sets V_p and V_q, and with partite sets V_r and V_s, and

joint hyperedges with partite sets V_q and V_r, and with partite sets V_s and V_p.

Since any pair of elements in $\{1,2,3,4\}$ occurs once in I and once in J, each pair of partite sets is used to construct inline hyperedges once and joint hyperedges once.

Note that the number of hyperedges of all cycles in $\mathscr{C}$ is $24m^2(2m-1)$ counted repeatedly. Since the number of hyperedges of Type 2 in $K^{(3)}_{4(2m)}$ is also $24m^2(2m-1)$, it suffices to show that each hyperedge of Type 2 is contained in at most one cycle in $\mathscr{C}$.

Let $e \in E(\mathcal{T}_2(K^{(3)}_{4(2m)}))$ be a hyperedge with $\langle x, y\rangle$-partite sets, say $e = \{a^x_u, a^x_v, a^y_d\}$. By the property of $\mathscr{D}$, x and y appear together in I or J once. By Lemma 2.9, e cannot be both inline hyperedge of a cycle and joint hyperedge of another cycle at the same time. Therefore, without loss of generality, there exists unique $D = (x, y, z, w) \in \mathscr{D}$ such that e is an inline hyperedge in $E(\{D\}, \mathscr{F})$.

Moreover, since $\mathscr{F}$ is a 1-factorization of K_{2m}, there exists unique $F = \{\{j, f(j)\} : j \in \{1, 2, \ldots, m\}\} \in \mathscr{F}$ such that $e \in \bigcup_{t=1}^{m} E(C_t(D, F))$.

To conclude that e is in at most one cycle, it suffices to show that inline hyperedges with $\langle x, y\rangle$-partite sets of the same length in $\bigcup_{t=1}^{m} E(C_t(D, F))$ are distinct. Let $\lambda \in \mathscr{L} \setminus \{0\}$. By Lemma 2.9$(i)$, since $C_0(D, F)$ has only one hyperedge of length λ with $\langle x, y\rangle$-partite sets, such hyperedge can be either $\{a^x_i, a^x_{f(i)}, a^y_{i-\lambda}\}$ or $\{a^x_i, a^x_{f(i)}, a^y_{f(i-\lambda)}\}$ for unique i. For $t \in \{1, 2, \ldots, m-1\}$, since $C_t(D, F)$ is the t^{th} rotation of $C_0(D, F)$, and the rotation preserves the lengths of hyperedges, the cycle $C_t(D, F)$ also contains exactly one hyperedge of length λ with $\langle x, y\rangle$-partite sets, namely $\{a^p_{i+t}, a^p_{f(i+t)}, a^q_{i-\lambda+t}\}$. Since $\{a^p_{i+t}, a^p_{f(i+t)}, a^q_{i-\lambda+t}\} \neq \{a^p_{i+w}, a^p_{f(i+w)}, a^q_{i-\lambda+w}\}$ if and only if $t \neq w$, all hyperedges of length λ are distinct. Hence each hyperedge is contained in at most one cycle in $\mathscr{C}$. Therefore $\mathscr{C}$ is a Hamiltonian decomposition of $\mathcal{T}_2(K^{(3)}_{4(2m)})$. $\square$

2.2.2 m is even

Let m be an even integer, say $m = 2\mu$. We have two initial cycles $C_0(D, F)$ and $\overline{C}_0(D, F)$, each of which is rotated which rotates $\mu - 1$ times to create additional $\mu - 1$ cycles. For $t \in \{0, 1, \ldots, \mu - 1\}$,

$$C_t(D, F) = (P_1^t \quad P_2^t) \quad \text{and} \quad \overline{C}_t(D, F) = (\overline{P}_1^t \quad \overline{P}_2^t)$$

where

$$(x,y) = \begin{cases} (p,q), & \text{if } j = 1, \\ (r,s), & \text{if } j = 2. \end{cases} \text{, and}$$

$$P_j^t = \begin{array}{llll} a^x_{1+t} & a^x_{f(1+t)} & a^y_{f(2\mu+t)} & a^y_{2\mu+t} \\ a^x_{2+t} & a^x_{f(2+t)} & a^y_{f(2\mu-1+t)} & a^y_{2\mu-1+t} \\ \vdots & \vdots & \vdots & \vdots \\ a^x_{\mu+t} & a^x_{f(\mu+t)} & a^y_{f(\mu+1+t)} & a^y_{\mu+1+t} \\ a^x_{\mu+1+t} & a^x_{f(\mu+1+t)} & a^y_{f(\mu+t)} & a^y_{\mu+t} \\ \vdots & \vdots & \vdots & \vdots \\ a^x_{2\mu-1+t} & a^x_{f(2\mu-1+t)} & a^y_{f(2+t)} & a^y_{2+t} \\ a^x_{2\mu+t} & a^x_{f(2\mu+t)} & a^y_{f(1+t)} & a^y_{1+t}, \end{array}$$

$$\overline{P}_j^t = \begin{array}{llll} a^x_{f(1+t)} & a^x_{1+t} & a^y_{2\mu+t} & a^y_{f(2\mu+t)} \\ a^x_{f(2+t)} & a^x_{2+t} & a^y_{2\mu-1+t} & a^y_{f(2\mu-1+t)} \\ \vdots & \vdots & \vdots & \vdots \\ a^x_{f(\mu+t)} & a^x_{\mu+t} & a^y_{\mu+1+t} & a^y_{f(\mu+1+t)} \\ a^x_{f(\mu+1+t)} & a^x_{\mu+1+t} & a^y_{\mu+t} & a^y_{f(\mu+t)} \\ \vdots & \vdots & \vdots & \vdots \\ a^x_{f(2\mu-1+t)} & a^x_{2\mu-1+t} & a^y_{2+t} & a^y_{f(2+t)} \\ a^x_{f(2\mu+t)} & a^x_{2\mu+t} & a^y_{1+t} & a^y_{f(1+t)}, \end{array}$$

We say that $C_t(D,F)$ and $\overline{C}_t(D,F)$ are the t^{th} *rotation* of $C_0(D,F)$ and $\overline{C}_0(D,F)$, respectively.

Example 2.11. *An illustration of the two initial cycles $C_0(D,F)$ and $\overline{C}_0(D,F)$ which are in the construction of a Hamiltonian decomposition of $K^{(3)}_{4(2m)}$ when $m = 6$, $D = (1,3,4,2)$ and $F = \{\{j, f(j)\} : j \in \{1,2,3,4,5,6\}\}$.* In the Figures 3(a) and 4(a), each vertex a^x_ℓ in the initial cycles $C_0(D,F)$ and $\overline{C}_0(D,F)$ is represented by its subscript ℓ. Moreover, Figures 3(b) and 4(b) illustrate the lengths of hyperedges. As each three consecutive vertices along the cycle form a hyperedge, we label its length at the middle vertex of such hyperedge.

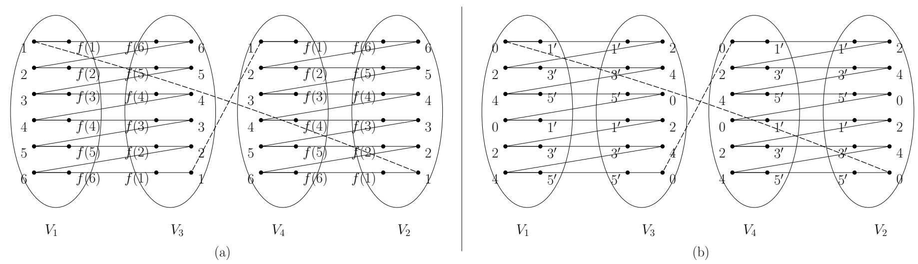

Figure 3: (a) $C_0(D,F)$ of $\mathcal{T}_2(K^{(3)}_{4(12)})$, and (b) the lengths of hyperedges in $C_0(D,F)$.

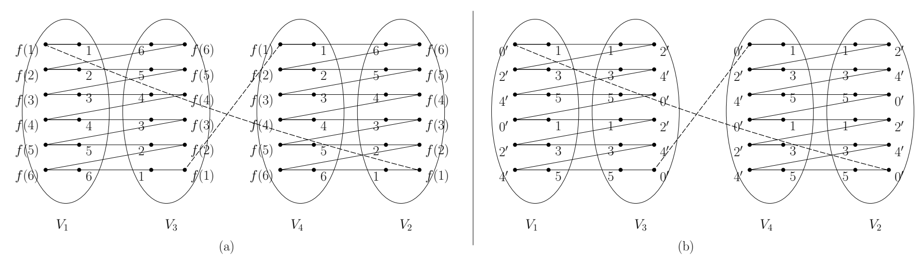

Figure 4: (a) $\overline{C}_0(D,F)$ of $\mathcal{T}_2(K^{(3)}_{4(12)})$, and (b) the lengths of hyperedges in $\overline{C}_0(D,F)$.

Theorem 2.12. *The subhypergraph $\mathcal{T}_2(K^{(3)}_{4(2m)})$ has a Hamiltonian decomposition when m is even.*

Proof. Let $m = 2\mu$, $\mathscr{D} = \{(1,2,3,4),(1,3,4,2),(1,4,2,3)\}$, $\mathscr{F}$ a 1-factorization of $K_{2m}([2m])$ and

$$\overline{\mathscr{C}} = \{C_t(D,F), \overline{C}_t(D,F) : t \in \{0,1,\ldots,\mu-1\}, D \in \mathscr{D}, \text{ and } F \in \mathscr{F}\}.$$

Let $D \in \mathscr{D}$ and $F \in \mathscr{F}$, written

$$D = (p,q,r,s) \text{ and } F = \{\{j, f(j)\} : j \in \{1,2,\ldots,m\}\}.$$

Similar to Lemma 2.4, the cycles $C_t(D,F)$ and $\overline{C}_t(D,F)$ constructed by D and F are Hamiltonian cycles of $\mathcal{T}_2(K^{(3)}_{4(2m)})$, thus $\overline{\mathscr{C}}$ is a collection of Hamiltonian cycles of $\mathcal{T}_2(K^{(3)}_{4(2m)})$. It remains to show that $\overline{\mathscr{C}}$ is a decomposition of $\mathcal{T}_2(K^{(3)}_{4(2m)})$.

We write $8m$ hyperedges in $E(C_0(D,F))$ and $8m$ hyperedges in $E(\overline{C}_0(D,F))$ in order around the cycles as $e_1, e_2, \ldots, e_{8m}$, and $\overline{e}_1, \overline{e}_2, \ldots, \overline{e}_{8m}$, respectively, beginning with

$$e_1 = \{a_1^p,\ a_{f(1)}^p, a_{f(2\mu)}^q\} \text{ and } e_2 = \{a_{f(1)}^p, a_{f(2\mu)}^q, a_{2\mu}^q\} \text{ and so on, and}$$

$$\overline{e}_1 = \{a_{f(1)}^p,\ a_1^p, a_{2\mu}^q\} \text{ and } \overline{e}_2 = \{a_1^p, a_{2\mu}^q, a_{f(2\mu)}^q\} \text{ and so on.}$$

Note that $C_0(D,F)$ is defined exactly the same as in Section 2.2.1, except even m. Besides, here we rotate $C_0(D,F)$ to construct additional $\frac{m}{2}-1$ cycles instead of $m-1$ cycles.

First we have that e_{2m} is an inline hyperedge of length 0 with $\langle p,q\rangle$-partite sets, and

e_{4m} is a joint hyperedge of length 0 with $\langle r,q\rangle$-partite sets.

For $\ell \equiv 0,1 \pmod 4$, $\ell \le 4m$ and $\ell \ne 2m, 4m$, e_ℓ is an inline hyperedge with $\langle p,q\rangle$-partite sets in $C_0(D,F)$ of length $\lambda \ne 0$. Tables 4 and 5 show such lengths.

ℓ	4	8	12	$\ldots$	$2m-4$	$2m$	$2m+4$	$2m+8$	$2m+12$	$\ldots$	$4m-4$	$4m$
Length of e_ℓ	2	4	6	$\ldots$	$m-2$	0	2	4	6	$\ldots$	$m-2$	0

Table 4: Lengths of e_ℓ where $\ell \equiv 0 \pmod 4$ and $\ell \le 4m$.

ℓ	1	5	9	$\ldots$	$2m-3$	$2m+1$	$2m+5$	$\ldots$	$4m-3$
Length of e_ℓ	$1'$	$3'$	$\ldots$	$(m-1)'$	$1'$	$3'$	$5'$	$\ldots$	$(m-1)'$

Table 5: Lengths of e_ℓ where $\ell \equiv 1 \pmod 4$ and $\ell \le 4m$.

Since m is even, the lengths of e_ℓ and $e_{2m+\ell}$ are the same for $\ell \in \{1,2,\ldots,2m\}$. In particular, for $\ell \in \{1,2,\ldots,2m-1\}$, $e_{2m+\ell}$ is the μ^{th} rotation of e_ℓ. Let $\mathscr{L}_1 = \{2,4,\ldots,m-\ 2\} \cup \{1',3'\ldots,(m-\ 1)'\}$. Therefore, the set of the lengths of $2m-2$ inline hyperedges with $\langle p,q\rangle$-partite sets in $C_0(D,F)$ is

$$\{\mathcal{L}_{(p,q)}(e_\ell)\colon \ell \equiv 0,1 \pmod 4, \ell \in \{0,1,\ldots,4m-2\} \setminus \{2m\}\} = 2\mathscr{L}_1.$$

For the lengths of e_ℓ where $\ell \equiv 2,3 \pmod 4$ and $\ell \le 4m$, we have the similar results for hyperedges with $\langle q,p\rangle$-partite sets and $\langle q,r\rangle$-partite sets as follows. e_{2m-1} and e_{4m-1} are hyperedges of length 0, and the set of lengths of $2m-2$ inline hyperedges with $\langle q,p\rangle$-partite sets in $C_0(D,F)$ is

$$\{\mathcal{L}_{(p,q)}(e_\ell)\colon \ell \equiv 2,3 \pmod 4, \ell \in \{0,1,\ldots,4m-2\} \setminus \{2m-1\}\} = 2\mathscr{L}_1.$$

Next, consider the lengths of hyperedges in $\overline{C}_0(D,F)$. Observe that $\overline{C}_t(D,F)$ is a modification of $C_t(D,F)$ by swapping a_i^x and $a_{f(i)}^x$ for all $x \in \{p,q,r,s\}$ and $i \in [m]$. Thus for $\ell \in \{1,2,\ldots,8m\}$ and $\lambda \in \{0,1,\ldots,m-1\}$,

$$\mathcal{L}(\overline{e}_\ell) = \lambda' \text{ if and only if } \mathcal{L}(e_\ell) = \lambda,$$

$$\mathcal{L}(\overline{e}_\ell) = \lambda \text{ if and only if } \mathcal{L}(e_\ell) = \lambda'.$$

Then the lengths of all joint hyperedges and inline hyperedges $\overline{e}_{2m-1}, \overline{e}_{2m}$, $\overline{e}_{2m-1}$ and $\overline{e}_{2m}$ are $0'$. Let $\mathscr{L}_2 = \{2',4',\ldots,(m-2)'\} \cup \{1,3\ldots,m-\ 1\}$. The remaining $2m-2$ inline hyperedges with $\langle p,q\rangle$-partite sets in $\overline{C}_0(D,F)$ have lengths spanning the multiset $2\mathscr{L}_2$ (See Tables 6 and 7). Also, $2m-2$ inline hyperedges with $\langle q,p\rangle$-partite sets have lengths spanning the multiset $2\mathscr{L}_2$.

ℓ	4	8	12	...	$2m-4$	$2m$	$2m+4$	$2m+8$	$2m+12$	...	$4m-4$	$4m$
Length of $\overline{e}_\ell$	$2'$	$4'$	$6'$	...	$(m-2)'$	$0'$	$2'$	$4'$	$6'$	...	$(m-2)'$	$0'$

Table 6: Lengths of $\overline{e}_\ell$ where $\ell \equiv 0 \pmod 4$ and $\ell \leq 4m$.

ℓ	1	5	9	...	$2m-7$	$2m-3$	$2m+1$	$2m+5$	...	$4m-3$
Length of $\overline{e}_\ell$	$1'$	$3'$	$5'$	...	$(m-1)'$	$1'$	$3'$	$5'$	...	$(m-1)'$

Table 7: Lengths of $\overline{e}_\ell$ where $\ell \equiv 1 \pmod 4$ and $\ell \leq 4m$.

Hence inline hyperedges with $\langle p,q\rangle$-partite sets (or $\langle q,p\rangle$-partite sets) in both $C_0(D,F)$ and $\overline{C}_0(D,F)$ except those of lengths 0 and $0'$ have lengths spanning the multiset $2\mathscr{L}_1 \cup 2\mathscr{L}_2$. Remark that the multiset $2\mathscr{L}_1 \cup 2\mathscr{L}_2 = 2\mathscr{L} \setminus\ 2\{0,0'\}$.

In conclusion, we have that for $\lambda_1 \in \mathscr{L}_1$, $C_0(D,F)$ contains exactly two hyperedges of length λ_1 with $\langle p,q\rangle$-partite sets, and for $\lambda_2 \in \mathscr{L}_2$, $\overline{C}_0(D,F)$ contains exactly two hyperedges of length λ_2 with $\langle p,q\rangle$-partite sets.

For $D \in \mathscr{D}$ and $F \in \mathscr{F}$, $E(D,F)$ stands for the collection of hyperedges of all cycles constructed by D and F. In other words, $E(D,F) = \bigcup_{t=0}^{\mu-1} E(C_t(D,F)) \cup \bigcup_{t=0}^{\mu-1} E(\overline{C}_t(D,F))$.

We will use the lengths of hyperedges in the cycles to prove that $\overline{\mathscr{C}}$ is the decomposition of $\mathcal{T}_2(K^{(3)}_{4(2m)})$. By the similar argument as in Theorem 2.10, we count the number of hyperedges of all cycles in $\overline{\mathscr{C}}$ and in $\mathcal{T}_2(K^{(3)}_{4(2m)})$. Then it suffices to show that any hyperedge of Type 2 with $\langle x,y\rangle$-partite sets, namely e, is contained in at most one cycle in $\overline{\mathscr{C}}$. From the proof of Theorem 2.10, the essential property of $\mathscr{D}$ implies that each pair of partite sets is used to construct inline hyperedges once and joint hyperedges once. Since $\mathscr{D}$ has such essential property and $\mathscr{F}$ is a 1-factorization, without loss of generality, there exists $D' = (x,y,z,w) \in \mathscr{D}$ and $F' = \{\{j, f'(j)\} : j \in \{1,2,\ldots,m\}\} \in \mathscr{F}$ such that $e \in E(D',F')$.

Since hyperedges in the same Hamiltonian cycle are always distinct, to conclude that e is in at most one cycle, we will claim that hyperedges with $\langle x,y\rangle$-partite sets of the same length in $E(D',F')$ are all distinct. The lengths of hyperedges in $C_0(D',F')$ and hyperedges in $\overline{C}_0(D',F')$ are in $\mathscr{L}_1 \cup \{0\}$ and $\mathscr{L}_2 \cup \{0'\}$, respectively. Since $\mathscr{L}_1 \cup \{0\}$ and $\mathscr{L}_2 \cup \{0'\}$ are disjoint, it is enough to show that hyperedges of the same length with $\langle x,y\rangle$-partite sets in $\bigcup_{t=0}^{\mu-1} E(C_t(D',F'))$ are distinct.

Let $\lambda \in \mathscr{L}_1 \backslash \{0\}$. Since $C_0(D',F')$ contains exactly two distinct hyperedges of length λ with $\langle x,y\rangle$-partite sets, such two hyperedges can be either a pair of hyperedges $e_1 = \{a^x_i, a^x_{f(i)},\ a^y_{i-\lambda}\}$, $e_2 = \{a^x_{i+\mu},\ a^x_{f(i+\mu)},\ a^y_{i+\mu-\lambda}\}$ or $e_1 = \{a^x_i, a^x_{f(i)}, a^y_{f(i-\lambda)}\}$, $e_2 = \{a^x_{i+\mu}, a^x_{f(i+\mu)},\ a^y_{f(i+\mu-\lambda)},\}$ for unique i. By the proof of Theorem 2.10, hyperedges of the same length obtained by the rotation are all distinct. Observe that we rotate each initial cycle in our construction at most $\mu - 1$ times. Although e_1 and e_2 are the μ^{th} rotation of each other, hyperedges obtained from rotating e_1 are different from hyperedges obtained from rotating e_2. Hence hyperedges with $\langle x,y\rangle$-partite sets of length λ in $\bigcup_{t=0}^{\mu-1} E(C_t(D',F'))$ are all distinct. Our claim holds. Thus any hyperedges of Type 2 is contained in at most one cycle in $\overline{\mathscr{C}}$, and therefore $\overline{\mathscr{C}}$ is a Hamiltonian decomposition of $\mathcal{T}_2(K^{(3)}_{4(2m)})$. □

Therefore by Theorems 2.2, 2.10 and 2.12, it can be concluded that $K^{(3)}_{4(2m)}$ has a Hamiltonian decomposition for all positive integer m.

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