***East-West J. of Mathematics: Vol. 15, No 1*** (2013) pp. 101-106

# VALUING DEFAULT RISK FOR ASSETS VALUE JUMP PROCESSES

**Hoang Thi Phuong Thao**

*Hanoi Univ. of Sciences, Vietnam National University*
*334 Nguyen Trai Road, Thanh Xuan, Hanoi, Vietnam*
*email: hoangthao09@gmail.com*

**Abstract**

The aim of this paper is to investigate the problem of valuing default risk for a firm when its assets value is a jumps-diffusion process.

## 1. Introduction

The quantitative model of risk initiated by Merton (1974) shows the probability of company default. In the classical Merton model, the firm asset value $V_t$ is given by

$$dV_t = \mu V_t dt + \sigma V_t dW_t, \tag{1.1}$$

where $\sigma$ is the asset volatility, $\mu$ is risk-free interest rate and $W_t$ is a Brownian motion (see [2]-[7]). This kind of models is considered under some risk-neutral probability.

In this paper we study the problem where $V_t$ is driven by jumps-diffusion consisting of a Brownian motion $W_t$ and a Poisson process $N_t$ of intensity $\lambda > 0$:

$$dV_t = \mu V_t dt + \sigma V_t dW_t + \gamma V_{t-} dN_t, \tag{1.2}$$

where $\mu$, $\sigma$, $\gamma$ are constant.

Firstly we recall about the solution of (1.2). It has been given explicitly without proof in [7] and we can see how to get it as follows.

We set

$$dX_t = \mu dt + \sigma dW_t + \gamma dN_t,$$

**Key words:** Default risk, jump processes, Merton model
2010 AMS Classification: 60H, 93E05.

then the (1.2) can be written as

$$dV_t = V_{t-}dX_t.$$

We denote by $X_t^c$ the continuous part of $X_t$ then $dX_t^c = \mu dt + \sigma dW_t$. It follows that

$$Y_t = \exp(\mu t - \frac{1}{2}\sigma^2 t + \sigma W_t)$$

is the solution of the equation

$$dY_t = Y_t dX_t^c = Y_{t-}dX_t^c.$$

Next, we define $J_t = 1$ for $t$ between 0 and the time of the first jump of $X$ and

$$J_t = \prod_{0 \leq s \leq t} (1 + \Delta X_s)$$

for $t$ greater than or equal to the first jump time of $X$. If $X$ has a jump at time $t$, then $J_t = J_{t-}(1 + \Delta X_t)$ and

$$\Delta J_t = J_t - J_{t-} = J_{t-}\Delta X_t = \gamma J_{t-}\Delta N_t = \gamma J_{t-}.$$

Therefore,

$$J_t = (1+\gamma)K_{t-} = (1+\gamma)^{N_t}.$$

Put $V_t = Y_t J_t$. By virtue of Ito's product rule and noting that $[Y, J](t) = 0$, $Y_{t-} = Y_t$ we get

$$\begin{aligned} dV_t &= Y_{t-}dJ_t + J_{t-}dY_t \\ &= Y_{t-}J_{t-}\Delta X_t + J_{t-}Y_{t-}dX_t^c \\ &= V_{t-}dX_t. \end{aligned}$$

Then solution of (1.2) is given by

$$V_t = Y_t J_t = V_0 \exp(\mu t - \frac{1}{2}\sigma^2 t + \sigma W_t)(1+\gamma)^{N_t}. \tag{1.3}$$

## 2. Default probability.

If at some time $t$ the asset's value of a company is less than its total debt $L$ that should be paid exactly at that time and the company has not ability to pay for this, it will jump into default state.

### 2.1 Default when $V_t$ is less than a liability $L$.

Assume that $V_t$ given by (1.2) and that $V_t < L$ at the time $t$, where $L$ denotes the total debt of the firm. Then we see from (1.3) that

$$V_t < L.$$

It follows from (1.3) that

$$\ln V_t = \ln V_0 + (\mu - \frac{\sigma^2}{2})t + \sigma W_t + N_t \ln(\gamma + 1) < \ln L. \tag{2.1}$$

And we have to find

$$P_{default} := P(\ln V_t < \ln L). \tag{2.2}$$

**Theorem 2.1.** *$P_{default}$ can be given by*

$$P_{default} = \sum_{n=0}^{\infty} \Phi(K - cn)\frac{(\lambda t)^n}{n!}e^{-\lambda t}, \tag{2.3}$$

*where $c = \frac{\ln(\gamma+1)}{\sigma\sqrt{t}}$, $K = \frac{\ln L - \ln V_0 - (\mu - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}$ and $\Phi$ is the standard normal distribution function.*

*Proof.* We have

$$\begin{aligned} P(\ln V_t < lnL) &= P(\sigma W_t + N_t \ln(\gamma+1) < \ln L - \ln V_0 - (\mu - \frac{1}{2}\sigma^2)t) \\ &= P(\frac{W_t}{\sqrt{t}} + \frac{\ln(\gamma+1)}{\sigma\sqrt{t}}N_t < \frac{\ln L - \ln V_0 - (\mu - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}) \\ &= P(Z_t + cN_t < K), \end{aligned} \tag{2.4}$$

where $Z_t = \frac{W_t}{\sqrt{t}}$ is standard normal, $c = \frac{\ln(\gamma+1)}{\sigma\sqrt{t}}$, $K = \frac{\ln L - \ln V_0 - (\mu - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}$ . We see that (2.4) is just a convolution of a Gaussian random variable and a Poisson random variable. And these two random variables are independent then

$$\begin{aligned} P(Z_t + cN_t < K) &= \sum_{k=0}^{\infty} P(N_t = k)P(Z_t + cN_t < K | N_t = k) \\ &= \sum_{k=0}^{\infty} P(N_t = k)P(Z_t + ck < K) \\ &= \sum_{k=0}^{\infty} P(Z_t + ck < K)\frac{(\lambda t)^k}{k!}e^{-\lambda t} \\ &= \sum_{k=0}^{\infty} P(Z_t < K - ck)\frac{(\lambda t)^k}{k!}e^{-\lambda t} \end{aligned} \tag{2.5}$$

Finally we have

$$P_{default} = P(Z_t + cN_t < K) = \sum_{k=0}^{\infty} \Phi(K - ck)\frac{(\lambda t)^k}{k!}e^{-\lambda t}$$

as stated. □

### 2.2. The case of many liabilities $L_1, L_2, ..., L_n$

Now we suppose that there are $n$ liabilities $L_1, L_2, ..., L_n$ that should be paid at times $t_1$, $t_2$,...,$t_n$ respectively, with $t_1 < t_2 < ... < t_n$.
Put $T = \max\{t_1, t_2, ..., t_n\} = t_n$. The company will jump into default position before the time $T$ if and only if at one of time $t_i$ $(i = 1, 2, ..., n)$, it happens that

$$V_{t_i} < L_i.$$

So the probability of default before $T$ is

$$P_{default}(0, T) = 1 - P(V_{t_i} > L_i, \forall t_i)$$

Denote $L = \max\{L_1, ..., L_n\}$ we easily see that

$$(V_{t_i} > L_i) \supset (V_{t_i} > L), \quad \forall t_i$$

Then

$$P_{default}(0, T) \leq 1 - P(V_{t_i} > L, \forall t_i). \tag{2.6}$$

Put $U_t = \sigma W_t + N_t \ln(1 + \gamma)$. The inequality $V_{t_i} > L$ is equivalent to

$$U_{t_i} = \sigma W_{t_i} + N_{t_i} \ln(\gamma + 1) > \ln L - \ln V_0 - (\mu - \frac{\sigma^2}{2})t_i := x_i.$$

Consider the event

$$A = \{V_{t_i} > L, \forall t_i\} = \bigcap_{i=1}^{n} \{U_{t_i} > x_i\}. \tag{2.7}$$

Since $W_t$ and $N_t$ are independent, moreover two processes $(W_t, t \geq 0)$, $(N_t, t \geq 0)$ are of independent increments then $(U_t)_t$ is of independent increment with $U_0 = 0$ a.s.
Denoting by $A_i$ the event $\{U_{t_i} > x_i\}, i = 1, 2, ..., n$ we can see that

$$A_1 = \{U_{t_1} > x_1\} = \{U_{t_1} - U_0 > x_1\},$$

$$A_2 = \{U_{t_2} > x_2\} = \{U_{t_2} - U_{t_1} > x_2 - U_{t_1}\} \supset \{U_{t_2} - U_{t_1} > x_2 - x_1\},$$

if $A_1$ occurs.

$$\cdots$$

$$A_n = \{U_{t_n} > x_n\} = \{U_{t_n} - U_{t_{n-1}} > x_n - U_{t_{n-1}}\} \supset \{U_{t_n} - U_{t_{n-1}} > x_n - x_{n-1}\},$$

if $A_1, ...A_{n-1}$ occur.

Put $B_i = \{U_{t_i} - U_{t_{i-1}} > x_i - x_{i-1}\}$ for $i = 1, 2, ..., n$ and $x_0 = 0$ by convention. It follows that

$$\bigcap_{i=1}^{n} B_i \subset \bigcap_{i=1}^{n} A_i = A.$$

Because of the independence of increments we have

$$P(A) \geq P(\bigcap_{i=1}^{n} B_i) = \prod_{i=1}^{n} P(B_i). \tag{2.8}$$

We have

$$\begin{aligned} P(B_i) &= P(U_{t_i} - U_{t_{i-1}} > x_i - x_{i-1}) = 1 - P(U_{t_i} - U_{t_{i-1}} \leq x_i - x_{i-1}) \\ &= 1 - P(\sigma(W_{t_i} - W_{t_{i-1}}) + (N_{t_i} - N_{t_{i-1}})\ln(\gamma + 1) < x_i - x_{i-1}) \\ &= 1 - P(\frac{(W_{t_i} - W_{t_{i-1}})}{\sqrt{t_i - t_{i-1}}} + \frac{N_{t_i} - N_{t_{i-1}}}{\sigma\sqrt{t_i - t_{i-1}}}\ln(\gamma + 1) < \frac{x_i - x_{i-1}}{\sigma\sqrt{t_i - t_{i-1}}}) \\ &= 1 - P(Z + d(N_{t_i} - N_{t_{i-1}}) < a), \end{aligned}$$

where $Z = \frac{(W_{t_i} - W_{t_{i-1}})}{\sqrt{t_i - t_{i-1}}}$ is of standard normal distribution $N(0,1)$, $d = \frac{\ln(\gamma+1)}{\sigma\sqrt{t_i - t_{i-1}}}$ and $M = \frac{x_i - x_{i-1}}{\sigma\sqrt{t_i - t_{i-1}}}$. So

$$\begin{aligned} P(B_i) &= 1 - \sum_{k=0}^{\infty} P(Z < M - dk)P(N_{t_i} - N_{t_{i-1}} = k) \\ &= 1 - \sum_{k=0}^{\infty} \Phi(M - dk)e^{-\lambda(t_i - t_{i-1})}\frac{(\lambda(t_i - t_{i-1}))^k}{k!}. \end{aligned} \tag{2.9}$$

Therefore

$$1 - P(A) \leq 1 - \prod_{i=1}^{n} P(B_i).$$

From (2.6), (2.7), (2.8) and (2.9) we have.

**Theorem 2.2** *The probability of default before $T$ is estimated by*

$$\begin{aligned} P_{default}(0,T) &\leq 1 - P(A) \\ &\leq 1 - \prod_{i=1}^{n}\left(1 - \sum_{k=0}^{\infty} \Phi(M - dk)e^{-\lambda(t_i - t_{i-1})}\frac{(\lambda(t_i - t_{i-1}))^k}{k!}\right), \end{aligned} \tag{2.10}$$

*where $x_i = \ln L - \ln V_0 - (\mu - \frac{\sigma^2}{2})t_i$, $d = \frac{\ln(\gamma+1)}{\sigma\sqrt{t_i - t_{i-1}}}$, $M = \frac{x_i - x_{i-1}}{\sigma\sqrt{t_i - t_{i-1}}}$ and $\Phi$ is the standard normal distribution function.*

# References


[1] T. Berg, *From actual to risk-neutral default probabilities: Merton and Beyond*, The Journal of Credit Risk, **6**(1), (2010), 55-86.

[2] T. Bielecki, M. Jeanblanc and M. Rutkowski, " Credit Risk Modeling", Center for Study of Insurance and Finance, Osaka University, 2009.

[3] H. Bystrom, *Merton for Dummies: A Flexible Way of Modelling Default Risk,* **112** (2007), Quantitative Finance Research Centre, University of Technology, Sydney, Research Paper Series.

[4] R. Cont and P. Tankov, "Financial Modelling With Jump Processes", Chapman and Hall, CRC Press, 2003.

[5] K. Giesecke and R. G. Lisa, *Forecasting Default in Face of Uncertainty,* The Journal of Derivatives, Fall (2004), 11-25.

[6] M. Roger, *Merton Robert C. on putting theory into practice,* CFA Magazine, **July-August** (2004), 34-37.

[7] S. R. Shreve , "Stochastic Calculus for Finance II", Springer, 2003.

[8] Y. Wang, *Structural Credit Risk Modeling: Merton and Beond*, Risk Managements, **June** (2009), 30-33.

[9] H. T. Phuong Thao and T. H. Thao, *A Note on A Model of Merton Type for Valuing Default Risk,* Applied Mathematical Sciences , **6** (89-92),(2012), 4457-4461.