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A NOTE ON FRACTIONAL KALMAN-BUCY FILTERING

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Abstract

A new approach to fractional Kalman-Bucy filtering is introduced, based on author's results on semimartingale L^2-approximation applied to fractional stochastics. Method of nonlinear filtering is used in the process of determining the filter.

1. Introduction

In this note we deal with the fractional Brownian motion of Liouville form (LfBm) B_t^H defined by

$$B_t^H = \int_0^t (t-s)^\alpha dW_s, \alpha = H - \frac{1}{2}$$

where H is a Hurst index which is supposed that $\frac{1}{2} < H < 1$ and W_t is a standard Brownian motion.

LfBm is the long memory component in the decomposition of fractional Brownian motion of Mandelbrot form W_t^H

$$W_t^H = C_H(Z_t + B_t^H)$$

where Z_t is a stochastic process having absolutely continuous trajectories and C_H is a constant depending only on H ([1]).

Any works of construction for a fractional stochastic calculus should face up a main difficulty that is the no-martingale property of fractional Brownian motion.

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Many contributions have been made for overcoming this difficulty ([1], [3], [4] and others). In [8,10] we have introduced a practical approach to fractional stochastics based on establishing some important facts as follows:

1. Consider the process $B_t^{(\epsilon)}$ defined by D. Nualart ([1])

$$B_t^{(\epsilon)} = \int_0^t (t-s+\epsilon)^\alpha dW_t, \text{ for any } \epsilon > 0, \alpha = H - 1/2,) \leq t \leq T.$$

   Then $B_t^{(\epsilon)}$ is a semimartingale:

$$dB_t^{(\epsilon)} = \alpha \varphi_t^\epsilon dt + \epsilon^\alpha dW_t$$

   where

$$\varphi_t^\epsilon = \int_0^t (t-s+\epsilon)^{\alpha-1} dW_s.$$

   We have prove in [8]:
   Theorem: *$B_t^{(\epsilon)}$ converges to B_t^H in $L^2(\Omega)$ as $\epsilon \to 0$ for any $H \in (0,1)$ and the convergence is uniform with respect to $t \in [0,T]$.*

2. Motivated by a formula of integration by part, we have defined the fractional integral of a process $f(t,\omega)$ as a L^2-limit:

$$\int_0^t f(s,\omega) dB_t^H := L^2 - \lim_{\epsilon \to 0} \int_0^t f(s,\omega) dB_s^{(\epsilon)}$$

   where the integral in the right hand side is with respect to a semimartingale and so well defined (refer to [10]).

3. A theorem of existence and uniqueness for solution of fractional stochastic differential equations driven by B_t^H has been established in this sense and some classes of fractional stochastic dynamical systems has been studied ([1]).

Turning back our attention to filtering problems, we will consider in this note a fractional version of Kalman-Bucy filtering, where signal and observation processes are driven by two different and independent LfBm's. By transforming the initial problem into consequent problems of filtering that are no more really linear, we will apply the method of nonlinear to obtain the final result.

2. Fractional Kalman-Bucy filtering problem.

Consider a fractional signal-observation system (X, Y) given by

$$dX_t = aX_t dt + dB_t^{H_1}, \ 0 \leq t \leq T \tag{2.1}$$

$$dY_t = bX_t + dB_t^{H_2},\ 0 \le t \le T \tag{2.2}$$

where a and b are constant, and

$$B_t^{H_1} = \int_0^t (t-s)^{\alpha_1} dW_t^{(1)},\ \alpha_1 = H_1 - 1/2,\ 1/2 < H_1 < 1$$

$$B_t^{H_2} = \int_0^t (t-s)^{\alpha_2} dW_t^{(2)},\ \alpha_2 = H_2 - 1/2,\ 1/2 < H_2 < 1$$

$W_t^{(1)}$ and $W_t^{(2)}$ are two independent standard Brownian motions.
The problem is to estimate $\widehat{X_t}$ under the observation Y_t:

$$\widehat{X_t} = E(X_t|\mathcal{F}_t^Y)$$

where $\mathcal{F}_t^Y$ is the observation $\sigma-$ field defined by

$$\mathcal{F}_t^Y = \sigma(Y_s, 0 \le s \le t).$$

Suppose always that X_t and Y_t are square-integrable processes: $E(X_t^2) < \infty$, $E(Y_t^2) < \infty,\ 0 \le t \le T$.

2.1 Main ideas for solving.

(a) Firstly we consider the approximation model

$$dX_t = aX_t dt + dB_t^{(\epsilon_1)} \tag{2.3}$$

$$dX_t = aX_t dt + dB_t^{(\epsilon_2)} \tag{2.4}$$

where

$$B_t^{(\epsilon_1)} = \int_0^t (t-s+\epsilon_1)^{\alpha_1} dW_t^{(1)}$$

$$B_t^{(\epsilon_2)} = \int_0^t (t-s+\epsilon_2)^{\alpha_2} dW_t^{(2)}$$

(b) Since

$$dB_t^{(\epsilon_1)} = \alpha_1 \varphi_t^{\epsilon_1} dt + \epsilon_1^{\alpha_1} dW_t^{(1)}$$

then

$$dX_t = (aX_t + \alpha_1 \varphi_t^{\epsilon_1})dt + \epsilon_1^{\alpha_1} dW_t^{(1)}$$

where $\varphi_t^{\epsilon_1} = \int_0^t (t+s+\epsilon_1)^{\alpha_1 - 1} dW_s^{(1)}$.
Put $H_t = (aX_t + \alpha_1 \varphi_t^{\epsilon_1})$.
Now the problem (2.3)-(2.4) becomes that of filtering for a semimartingale

$$X_t = \int_0^t H_s ds + \epsilon^{\alpha_1} dW_t^{(1)} \tag{2.5}$$

with the observation given by (2.4) perturbed by the noise $dB_t^{(\epsilon_2)}$.

(c) Since $dB_t^{(\epsilon_2)} = \alpha_2 \varphi_t^{\epsilon_2} dt + \epsilon_2^{\alpha_2} dW_t^{(2)}$, (2.4) can be rewritten as

$$dY_t = (bX_t + \alpha_2 \varphi_t^{\epsilon_2})dt + \epsilon_2^{\alpha_2} dW_t^{(2)}$$

Put

$$h_t = (bX_t + \alpha_2 \varphi_t^{\epsilon_2}) \tag{*}$$

the we have the filtering problem

$$\text{Signal: } X_t = \int_0^t H_s ds + \epsilon_1^{\alpha_1} W_t^{(1)} \tag{2.5}$$

$$\text{Observation: } Y_t = \int_0^t h_s ds + \epsilon_2^{\alpha_2} W_t^{(2)} \tag{2.6}$$

As being shown later we can see that $E \int_0^t H_s^2 ds < \infty$ and $E \int_0^t h_s^2 ds < \infty$. Denote the filter for this problem by $\widehat{X_t^{(\epsilon_1,\epsilon_2)}}$. We will prove that the process

$$\widehat{X_t^{\epsilon_1}} := L^2 - \lim_{\epsilon_2 \to 0} \widehat{X_t^{(\epsilon_1,\epsilon_2)}} \tag{2.7}$$

is exactly the solution of the filtering problem

$$\text{Signal: } X_t^{\epsilon_1} = \int_0^t H_s ds + \epsilon_1^{\alpha_1} W_t^{(1)} \tag{2.8}$$

$$\text{Fractional observation: } Y_t = bX_t dt + dB_t^{H_2} \tag{2.9}$$

(d) Finally, denoting by $\widehat{X_t}$ the solution for the initial problem (2.1)-(2.2), we will prove that

$$\widehat{X_t} = L^2 - \lim_{\epsilon_1 \to 0} \widehat{X_t^{\epsilon_1}} = L^2 - \lim_{\epsilon_1 \to 0} (L^2 - \lim_{\epsilon_2 \to 0} \widehat{X_t^{\epsilon_1,\epsilon_2}}) \tag{2.10}$$

2.2. Solving problem (2.8)-(2.9).

2.2.1 We begin with the proof for (2.7).

We take $\epsilon_2 = 1/n$ and put

$\mathcal{F}_t^Y$: $\sigma-$algebra generated by $(Y_s, s \leq t)$, Y_t given by (2.2).

$\mathcal{F}_t^{Y^{1/n}}$: $\sigma-$algebra generated by $(Y_s^{1/n}, s \leq t)$,where

$$Y_t^{1/n} = \int_0^t h_s ds + \frac{1}{n^{\alpha_2}} W_t^{(2)}$$

The filter $\widehat{X_t^{\epsilon_1}}$ for system (2.5)-(2.2) is defined by

$$\widehat{X_t^{\epsilon_1}} = E(X_1^{\epsilon_1}|\mathcal{F}_t^Y),$$

where $X_t^{\epsilon_1}$ is given by (2.8).
Denote also by $\widehat{X_t^{\epsilon_1}}^{(n)}$ the filter of $X_t^{\epsilon_1}$ based on the observation $Y_t^{1/n}$:

$$\widehat{X_t^{\epsilon_1}}^{(n)} = E(X_t^{\epsilon_1}|\mathcal{F}_t^{Y^{1/n}})$$

Theorem 2.1. *The filter* $\widehat{X_t^{\epsilon_1}}^{(n)}$ *converges to* $\widehat{X_t^{\epsilon_1}}$ *as* $n \to \infty$.

Proof (refer to [8]). Consider two observations

$$Y_t^{1/n} = \int_0^t h_s ds + \frac{1}{n^{\alpha_2}} W_t^{(2)} \tag{2.11}$$

$$Z_t^{1/n} = \int_0^t h_s ds + \frac{1}{n^{\alpha_2}} W_{t+1/n}^{(2)} \tag{2.12}$$

We have

$$E|Y_t^{1/n} - Z_t^{1/n}|^2 = \frac{1}{n^{2\alpha_2}} E|W_t^{(2)} - W_{t+1/n}^{(2)}|^2 = \frac{1}{n^{2\alpha_2}}\frac{1}{n} = \frac{1}{n^{2\alpha_2+1}} \to 0$$

as $n \to \infty$.
Now it follows from the convergence

$$||Y_t^{1/n} - Z_t^{1/n}||_{L^2} \to 0 \text{ as } n \to \infty$$

that

$$||E(X_t^{\epsilon_1}|Y_t^{1/n}) - E(X_t^{\epsilon_1}|Z_t^{1/n})|| \to 0 \text{ as } n \to \infty$$

or more general

$$||E(X_t^{\epsilon_1}|\mathcal{F}_t^{Y^{1/n}}) - E(X_t^{\epsilon_1}|\mathcal{F}_t^{Z^{1/n}})|| \to 0 \text{ as } n \to \infty \tag{2.13}$$

where $\mathcal{F}_t^{Z^{1/n}} = \sigma(Z_s^{1/n}, 0 \le s \le t)$. Because the family $(\mathcal{F}_t^{Z^{1/n}})_n$ is non-increasing such that $\cap_n \mathcal{F}_t^{Z^{1/n}} = \mathcal{F}_t^Y$ then we see from a theorem on the convergence of conditional expectations that (refer to [6,p. 409]):

$$E(X_t^{\epsilon_1}|\mathcal{F}_t^{Z^{1/n}}) \longrightarrow_{L^2} E(X_t^{\epsilon_1}|\mathcal{F}_t^Y), n \to \infty \tag{2.14}$$

A combination of (2.13)and (2.14) yields

$$E(X_t^{\epsilon_1}|\mathcal{F}_t^{Y^{1/n}}) \longrightarrow_{L^2} E(X_t^{\epsilon_1}|\mathcal{F}_t^Y), n \to \infty$$

or

$$\widehat{X_t^{\epsilon_1}}^{(n)} \longrightarrow_{L^2} \widehat{X_t^{\epsilon_1}},\ n \to \infty$$

that is another form of (2.7), where $\widehat{X_t^{\epsilon_1}}^{(n)} = \widehat{X_t^{(\epsilon_1,\epsilon_2)}}$. □

2.2.2. Calculation of $\widehat{X_t^{\epsilon_1}}^{(n)}$, the filter for system (2.5)-(2.6).
This system can be rewritten as

$$X_t = \int_0^t H_s ds + \epsilon_1^{\alpha_1} W_t^{(1)}, \tag{2.16}$$

$$Y_t^{1/n} = \int_0^t h_s ds + \frac{1}{n^{\alpha_2}} W_t^{(2)} \tag{2.17}$$

We have to check the condition

$$E \int_0^t h_s^2 ds < \infty \text{ for all } t \in [0,T] \tag{2.18}$$

Indeed, it follows from (*) and (2.17) that

$$h_t = bX_t + \alpha_2 \varphi_t^{1/n}$$

and

$$Y_t^{1/n} = \int_0^t (bX_s + \alpha_2 \varphi_s^{1/n}) ds + \frac{1}{n^{\alpha_2}} W_t^{(2)},\ 0 \le t \le T \tag{2.17'}$$

We have

$$h_s^2 \le 2(b^2 X_s^2 + \alpha_2^2 (\varphi_s^{1/n})^2)$$
$$Eh_s^2 \le 2(b^2 E X_s^2 + \alpha_2^2 E(\varphi_s^{1/n})^2)$$

By the Ito isometry we see that

$$E(\varphi_s^{1/n})^2 = E[(\int_0^t (t-s+\frac{1}{n})^{\alpha_2-1} dW_s)^2]$$
$$= \int_0^t (t-s+\frac{1}{n})^{2\alpha_2-2} ds \le \int_0^T (t-s+\frac{1}{n})^{2\alpha_2-2} ds < \infty$$

It is clear that

$$\int_0^t h_s^2 ds \le 2(b^2 \int_0^t X_s^2 ds + \alpha_2^2 \int_0^t (\varphi_s^{1/n})^2 ds)$$

and then

$$E \int_0^t h_s^2 ds \le 2(b^2 \int_0^t E X_s^2 ds + \alpha_2^2 \int_0^t E(\varphi_s^{1/n})^2 ds) < \infty.$$

Similarly we have also $E\int_0^t H_s^2 ds < \infty$.
Now we define the innovation process

$$I_t^{1/n} = Y_t^{1/n} - \int_0^t \widehat{h_s}^{(n)} ds \tag{2.19}$$

where

$$\widehat{h_s}^{(n)} = E(h_t|\mathcal{F}_t^{Y^{1/n}})$$

then $I_t^{1/n}$ is a $\mathcal{F}_t^{Y^{1/n}}$ - martingale.
Finally, we can write down the FKK (Fujisaki - Kallianpur - Kunita) equation for the filter $\widehat{X_t^{\epsilon_1}}^{(n)}$ of the system (2.16)-(2.17) as stated in the following theorem, where for the sake of simplification of notation we denote by $\pi_t^{(n)}(.)$ the filter $E(.|\mathcal{F}_t^{Y^{1/n}})$, for example $\pi_t^{(n)}(X_t^{\epsilon_1}) = E(X_t^{\epsilon_1}|\mathcal{F}_t^{Y^{1/n}})$.

Theorem 2.2.

$$\widehat{X_t^{\epsilon_1}}^{(n)} = \pi_t^{(n)}(X^{\epsilon_1}) = \int_0^t \pi_s^{(n)}(X^{\epsilon_1}.H)ds + \\ + \int_0^t [\pi_s^{(n)}(X^{\epsilon_1}.h) - \pi_s^{(n)}(X^{\epsilon_1})\pi_s^{(n)}(h)]dI_s \tag{2.20}$$

where $H_s = aX_s^{\epsilon_1} + \alpha_1\varphi_s^{\epsilon_1}$, $h_s = bX_s^{\epsilon_1} + \frac{1}{n^{\alpha_2}}\varphi_s^{1/n}$ *and* I_t *is the innovation process.*

Up to now, by Theorem 2.1 and Theorem 2.2 we can conclude that the filter $\widehat{X_t^{\epsilon_1}}$ of the system (2.8)-(2.9) is given by

$$\widehat{X_t^{\epsilon_1}} = L^2 - \lim_{n\to\infty} \widehat{X_t^{\epsilon_1}}^{(n)} \tag{2.21}$$

where $\widehat{X_t^{\epsilon_1}}^{(n)}$ satisfies the filtering equation (2.20).

3. Filtering for system (2.1)-(2.2)

Now we rewrite two equations (2.1) and (2.3) as follows

$$dX_t = aX_t dt + dB_t^{H_1} \tag{2.1}$$

$$dX_t^{\epsilon_1} = aX_t^{\epsilon_1} dt + dB_t^{\epsilon_1} \tag{2.3'}$$

where

$$B_t^{H_1} = \int_0^t (t-s)^{\alpha_1} dW_t^{(1)},\ \alpha_1 = H_1 - 1/2$$

$$B_t^{(\epsilon_1)} = \int_0^t (t-s+\epsilon_1)^{\alpha_1} dW_t^{(1)},$$

Theorem 3.1. *The solution $X_t^{\epsilon_1}$ of (2.3') converges to the solution X_t of (2.1) as $\epsilon_1 \to 0$ and the converges is uniform w.r.t. $t \in [0,T]$.*

Proof We get from (2.1) and (2.3')

$$X_t - X_t^{\epsilon_1} = a\int_0^t (X_s - X_s^{\epsilon_1})ds + B_t^{H_1} - B_t^{(\epsilon_1)} \tag{3.1}$$

Noticing that $||B_t^{H_1} - B_t^{(\epsilon_1)}|| \leq C(\alpha_1)\epsilon^{\alpha_1+\frac{1}{2}}$ as recalled in Section 1 then we have

$$||X_t - X_t^{\epsilon_1}|| \leq |a|\int_0^t ||X_s - X_s^{\epsilon_1}||ds + C(\alpha_1)\epsilon^{\alpha_1+\frac{1}{2}} \tag{3.2}$$

where $||.||$ stands for L^2-norm.
An application of Gronwall's lemma to (3.2) yields:

$$||X_t - X_t^{\epsilon_1}|| \leq C(\alpha_1)\epsilon^{\alpha_1+1/2}e^{t|a|} \tag{3.3}$$

It follows that

$$\sup_{0\leq t\leq T} ||X_t - X_t^{\epsilon_1}|| \leq C(\alpha_1)\epsilon^{\alpha_1+1/2}e^{t|a|} \tag{3.4}$$

Then $X_t^{\epsilon_1} \longrightarrow X_t$ in L^2 uniformly with respect to $t \in [0,T]$ □

Remark. In fact, (2.1) is a *fractional stochastic Langevin equation* and its solution is called *a fractional Ornstein-Uhlenbeck* given by

$$X_t = X_0 e^{at} + \int_0^t e^{a(t-s)} dB_s^{H_1} \tag{3.5}$$

(refer to [11]).
Next, we take $\epsilon_1 = 1/m$, $m \in \mathbb{Z}$ and redenote $X_t^{\epsilon_1}$ now by $X_t^{(m)}$. For each m we consider the system

$$dX_t^{(m)} = H_t dt + \frac{1}{m^{\alpha_1}} dW_t^{(1)} \tag{2.8'}$$

$$dY_t = bX_t^{(m)}dt + dB_t^{H_2} \tag{2.9'}$$

and we have already the filter $\widehat{X_t^{(m)}}$ given by (2.21):

$$\widehat{X_t^{(m)}} = L^2 - \lim_{n\to\infty} \widehat{X_t^{(m)}}^{(n)} \tag{3.6}$$

Proposition 3.1. $\widehat{X_t^{(m)}}^{(n)}$ *converges to required filter* $\widehat{X_t}$ *in* $L^2(\Omega, \mathcal{F}, P)$ *as* $m \to \infty$, $n \to \infty$.

Proof By definition $\mathcal{F}_t^Y = \lim_{n\to\infty} \mathcal{F}_t^{Y^{1/n}}$ where $(\mathcal{F}_t^{Y^{1/n}})_n$ is a non-increasing family of $\sigma-$fields.
We have now

$$\widehat{X_t^{(m)}} \longrightarrow_{L^2} X_t,\ m \to \infty \tag{3.7}$$

and

$$\mathcal{F}_t^{Y^{1/n}} \searrow \mathcal{F}_t^Y,\ n \to \infty \tag{3.8}$$

then by a result of convergence of conditional expectations we have

$$E(X_t^{(m)}|\mathcal{F}_t^{Y^{1/n}}) \longrightarrow_{L^2} E(X_t|\mathcal{F}_t^Y) \text{ as } m, n \to \infty \tag{3.9}$$

or

$$\widehat{X_t^{\epsilon_1}}^{(n)} \longrightarrow_{L^2} \widehat{X_t} \text{ as } \epsilon_1 = \frac{1}{m} \to 0 \text{ and } n \to \infty$$

or by other notation

$$\pi_t^{(n)}(X^{(m)}) \longrightarrow_{L^2} \widehat{X_t} \text{ as } m, n \to \infty,$$

where $\pi_t^{(n)}(X^{(m)}) = \pi^{(n)}(X_t^{\epsilon_1})$ satisfies Equation (2.20).
□

Conclusion. The filter for the initial system (X, Y) satisfying (2.1)-(2.2) is a process $\widehat{X_t}$ determined by

$$\widehat{X_t} = L^2 - \lim_{m,n\to\infty} \pi_t^{(n)}(X^{(m)}) \tag{3.10}$$

where $\pi_t^{(n)}(X^{(m)}) = \pi_t^{(n)}(X^{\epsilon_1})$ is solution of the filtering equation (2.20).

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