East-West J. of Mathematics: Vol. 15, No 2 (2013) pp. 152-157

CLIQUE-CHROMATIC NUMBERS OF CLAW-FREE GRAPHS

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Abstract

The clique-chromatic number of a graph is the least number of colors on the vertices of the graph so that no maximal clique of size at least two is monochromatic. A well-known result proved by Gravier et al. in 2003 suggests that the family of claw-free graphs has no bounded clique-chromatic number. Basco et al. explored more in 2004 that the family of claw-free graphs without odd holes has a bounded clique-chromatic number, in particular, these graphs are 2-clique-colorable. In this paper, we study some other subclasses of the family of claw-free graphs with a bounded clique-chromatic number, namely, claw-free graphs without an induced paw and claw-free graphs without an induced diamond.

1 Introduction

All graphs considered in this paper are simple. We use terminologies from West's textbook [12]. The vertex set of a graph G is denoted by $V(G)$. The symbols K_n, P_n and C_n denote the complete graph, path, and cycle, with n vertices, respectively. The *neighborhood* of a vertex x in a graph G is the set of vertices adjacent to x, and is denoted by $N_G(x)$. A subgraph H of a graph G is said to be *induced* if, for any pair of vertices x and y of H, xy is an edge of

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†Partially supported by His Royal Highness Crown Prince Maha Vajiralongkorn Fund
Key words: clique-chromatic number, clique-coloring, claw-free graph.
2010 Mathematics Subject Classification: 05C15

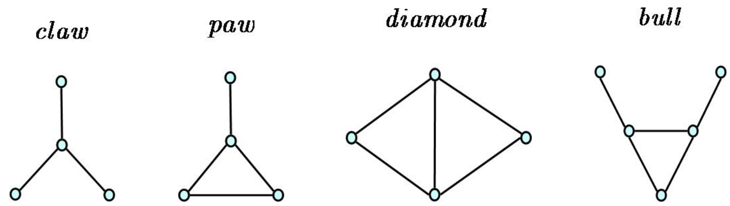
H if and only if xy is an edge of G. If an induced subgraph H is chosen based on a vertex subset S of $V(G)$, then H can be written as $G[S]$ and is said to be *induced by S*. A subset Q of $V(G)$ is a *clique* of G if any two vertices of Q are adjacent. A clique is *maximal* if it is not properly contained in another clique. A *k-coloring* of a graph G is a function $f : V(G) \to \{1, 2, ..., k\}$. A *proper k-coloring* of a graph G is a k-coloring of G such that adjacent vertices have different colors. The *chromatic number* of a graph G is the smallest positive integer k such that G has a proper k-coloring, denoted by $\chi(G)$. A *proper k-clique-coloring* of a graph G is a k-coloring of G such that no maximal clique of G with size at least two is monochromatic. A graph G is *k-clique-colorable* if G has a proper k-clique-coloring. The *clique-chromatic number* of G is the smallest k such that G has a proper k-clique-coloring, denoted by $\chi_c(G)$.

Note that $\chi_c(G) = 1$ if and only if G is an edgeless graph. Throughout this paper, a graph has at least one edge. Since any proper k-coloring of G is a proper k-clique-coloring of G, $\chi_c(G) \leq \chi(G)$. Recall that a *triangle* is the complete graph K_3. If G is a triangle-free graph, then maximal cliques of G are edges, so $\chi_c(G) = \chi(G)$. In 1955, Mycielski [8] showed that the family of triangle-free graphs has no bounded chromatic number. Consequently, it has no bounded clique-chromatic number, either. On the other hand, some families of graphs have bounded clique-chromatic numbers, for example, comparability graphs, cocomparability graphs, and the k-power of cycles (see [2], [4] and [5]). In 2004, Bacso et al. [1] showed that almost all perfect graphs are 3-clique-colorable and conjectured that all perfect graphs are 3-clique-colorable.

For a given graph F, a graph G is *F-free* if it does not contain F as an induced subgraph. A graph G is *$(F_1, F_2, ..., F_k)$-free* if it is F_i-free for all $1 \leq i \leq k$. Many authors explored more results in $(F_1, F_2, ..., F_k)$-free graphs. In 2003, Gravier, Hoang and Maffray [6] gave a significant result that, for any graph F, the family of F-free graphs has a bounded clique-chromatic number if and only if F is a vertex-disjoint union of paths. In [7], Gravier and Skrekovski proved that $(P_3 + P_1)$-free graphs unless it is C_5, and (P_5, C_5)-free graphs are 2-clique-colorable.

Recall that a *claw* is the complete bipartite graph $K_{1,3}$. A *paw* is the claw plus an edge, and a *diamond* is the complete graph K_4 minus an edge. In 2004, Bacso et al. [1] proved that (claw, odd hole)-free graphs are 2-clique-colorable. Later, Defossez in 2006 [3] showed that (diamond, odd hole)-free graphs are 4-clique-colorable and (bull, odd hole)-free graphs are 2-clique-colorable.

Since a claw is not a vertex-disjoint union of paths, by the result of Gravier et al. [6], the family of claw-free graphs has no bounded clique-chromatic number. In this paper, we focus on some subclasses of the family of claw-free graphs with a bounded clique-chromatic number.



2 (Claw, paw)-free graphs

The characterization of paw-free graphs in Theorem 1 proved by Olariu [9] is useful to prove our main result in Theorem 4.

Theorem 1. [9] *If G is a paw-free graph, then each component of G is either triangle-free or complete multipartite.*

Lemma 2. *Let G be a complete multipartite graph with at least one edge. Then $\chi_c(G) = 2$.*

Proof. Since each maximal clique of G intersects every partite set of G, labeling all vertices of one partite set of G by color 1 and the remaining vertices by color 2 provides a proper 2-clique-coloring of G. So $\chi_c(G) = 2$. □

Lemma 3. *Let G be a (claw, triangle)-free graph. Then each component of G is a path or a cycle.*

Proof. Let H be a component of G. If H contains no cycle, then H is a tree. Since H is claw-free, H is a path. Now, assume that H contains an induced cycle C. Suppose $H \neq C$. Then there exists a vertex v outside C which is adjacent to some vertex u in C. Since neighborhoods of u in C are not adjacent and H is claw-free, one of them, say w, must be adjacent to v. Then $\{u, v, w\}$ forms a triangle in H, a contradiction. Hence H is a cycle. □

Recall that a *hole* in a graph is an induced cycle with at least four vertices. An *odd (even) hole* is a hole with an odd (even, respectively) number of vertices.

Theorem 4. *Let G be a (claw, paw)-free graph with at least one edge. Then*

$$\chi_c(G) = \begin{cases} 2 & \textit{if } G \textit{ has no odd hole component,} \\ 3 & \textit{otherwise.} \end{cases}$$

Proof. Without lost of generality, assume that G is connected. Since G is paw-free, by Theorem 1, G is either triangle-free or complete multipartite. If G is complete multipartite, then $\chi_c(G) = 2$ by Lemma 2. Now, assume that G is triangle-free. Then G is (claw, triangle)-free. By Lemma 3, G is a path or a

cycle. If G is an odd cycle with at least five vertices, then $\chi_c(G) = \chi(G) = 3$. Hence $\chi_c(G) = 2$ if and only if G is not an odd cycle with at least five vertices.
□

3 (Claw, diamond)-free graphs

It is unknown whether the family of all (claw, diamond)-free graphs has a bounded clique-chromatic number. In this section, we introduce two subfamilies
of (claw, diamond)-free graphs having bounded clique-chromatic numbers, namely, (claw, diamond)-free graphs without even holes, and (claw, diamond)-free graphs without maximal cliques of size three.

Lemma 5. *Let x be a vertex in a diamond-free graph G. Then $N_G(x)$ is a disjoint union of cliques of G.*

Proof. Let H be a component of $G[N_G(x)]$. Suppose that $V(H)$ is not a clique of G. Then there are non-adjacent vertices a and b in H. Since H is connected, there is a path P between a and b. It follows that P contains an induced path P_3 of G. Then such induced path P_3 and the vertex x form an induced diamond of G, a contradiction. Hence $V(H)$ is clique of G. □

Lemma 6. *Let G be a connected (claw, diamond, even hole)-free graph. If G has a vertex contained in only one maximal clique of G, then G is 2-clique-colorable.*

Proof. Let x be a vertex contained in only one maximal clique of G. Define $A_0 = \{x\}$, $A_1 = N_G(x)$, and $A_i = N_G(A_{i-1}) \setminus (A_{i-1} \cup A_{i-2})$ for all $i \geq 2$. Then $V(G) = \bigcup_i A_i$. Note that A_1 is a clique of G. Define a coloring of G by labeling the vertices of A_i by color 1 if i is even, and by color 2 if i is odd.

Suppose that this coloring yields a monocolored maximal clique Q of size at least two. Then $Q \subseteq A_i$ for some $i \geq 2$. Let $u_i, v_i \in Q$. Then there is a vertex u_{i-1} in A_{i-1} which is adjacent to u_i. Suppose that u_{i-1} is adjacent to v_i. Since Q is a maximal clique of G, there is a vertex w in Q which is not adjacent to u_{i-1}. Then $\{u_{i-1}, u_i, v_i, w\}$ induces a diamond, a contradiction. So u_{i-1} is not adjacent to v_i. Similarly, there is a vertex v_{i-1} in A_{i-1} which is adjacent to v_i but not to u_i.

Since G is C_4-free, u_{i-1} cannot be adjacent to v_{i-1}. So $i \geq 3$. Let $u_{i-2}, v_{i-2} \in A_{i-2}$ such that u_{i-2} is adjacent to u_{i-1} and v_{i-2} is adjacent to v_{i-1}. If $u_{i-2} = v_{i-2}$, then there is a vertex u_{i-3} in A_{i-3} which is adjacent to u_{i-2}, and it follows that $\{u_{i-3}, u_{i-2}, u_{i-1}, v_{i-1}\}$ induces a claw, a contradiction. Thus $u_{i-2} \neq v_{i-2}$. Since G is claw-free, u_{i-2} is not adjacent to v_{i-1} and v_{i-2} is not adjacent to u_{i-1}. Since G is C_6-free, u_{i-2} is not adjacent to

v_{i-2}. Continue this way until we have $u_1, v_1 \in A_1$. Since A_1 is a clique, we eventually have an even hole, a contradiction. Hence this coloring is a proper 2-clique-coloring of G. □

Theorem 7. *Every (claw, diamond, even hole)-free graph is 3-clique-colorable.*

Proof. Let G be a (claw, diamond, even hole)-free graph. Without lost of generality, assume that G is connected. Let $x \in V(G)$. By Lemma 5, $N_G(x)$ is a disjoint union of r cliques of G for some integer r. Since G is claw-free, $r \leq 2$. If $r = 1$, then the theorem is proved by Lemma 6. Now, let $N_G(x) = A_1 \cup B_1$ where A_1 and B_1 are cliques of G. Define $A_i = N_G(A_{i-1}) \setminus (A_{i-1} \cup A_{i-2})$ and $B_i = N_G(B_{i-1}) \setminus (B_{i-1} \cup B_{i-2})$ for all $i \geq 2$. Then $V(G) = \{x\} \cup (\bigcup_i A_i) \cup (\bigcup_j B_j)$.

Case 1: $(\bigcup_i A_i) \cap (\bigcup_j B_j) = \phi$. By Lemma 6, both of $G[(\bigcup_i A_i) \cup \{x\}]$ and $G[(\bigcup_j B_j) \cup \{x\}]$ have a proper 2-clique-coloring. Combining these two colorings by identifying the color of x yields a proper 2-clique-coloring of G, so G is 2-clique-colorable.

Case 2: $(\bigcup_i A_i) \cap (\bigcup_j B_j) \neq \phi$. Let G' be the subgraph of G obtained by deleting all vertices of B_1. Then G' is a connected (claw, diamond, even hole)-free graph with x satisfying the condition in Lemma 6. Thus G' has a proper 2-clique-coloring. We can extend this coloring to a proper 3-clique-coloring of G by labeling color 3 to all vertices of B_1, and hence G is 3-clique-colorable. □

Note that all odd cycles C_{2n+1} ($n \geq 2$) are (claw, diamond, even hole)-free and $\chi_c(C_{2n+1}) = 3$. Thus the upper bound in Theorem 7 is sharp.

Now, we focus on (claw, diamond)-free graphs without maximal cliques of size three. The *line graph* of a graph G, written $L(G)$, is the graph whose vertices are the edges of G; and for any edges e and f in G, ef is an edge in $L(G)$ if and only if e and f have a common endpoint in G. A graph G is a *line graph* if there is a simple graph H such that $L(H) = G$. Let T be a triangle in a graph G. We say that T is *odd* if $|N_G(v) \cap V(T)|$ is odd for some $v \in V(G)$. In [10], van Rooij and Wilf proved that a graph G is a line graph if and only if G is claw-free and no induced diamond of G has two odd triangles. Hence all (claw, diamond)-free graphs are line graphs. Moreover, the clique-chromatic numbers of line graphs of triangle-free graphs is characterized in [11], as follows:

Theorem 8. [11] *Let H be a triangle-free graph. Then $\chi_c(L(H)) \leq 3$. Furthermore, $L(H)$ is 2-clique-colorable if and only if H has no odd hole component.*

The next corollary gives the characterization of the clique-chromatic numbers of (claw, diamond)-free graphs without maximal cliques of size three.

Corollary 9. *Let G be a (claw, diamond)-free graph with at least one edge. If G has no maximal clique of size three, then*

$$\chi_c(G) = \begin{cases} 2 & \textit{if } G \textit{ has no odd hole component,} \\ 3 & \textit{otherwise.} \end{cases}$$

Proof. Since G is a line graph, there is a simple graph H such that $G = L(H)$. If H has a triangle T, then T corresponds to a maximal clique of size three in $L(H) = G$, a contradiction. Thus H is triangle-free. Then the corollary follows directly from Theorem 8 and the fact that G has an odd hole component if and only if H has an odd hole component. □

Acknowledgement The authors would like to thank Prof. Narong Punnim for his valuable suggestions and comments.

References

[1] G. Bacsó, S. Gravier, A. Gyárfás, M. Preissmann and A. Sebő, Coloring the maximal cliques of graphs, *SIAM J. Discrete Math.*, **17** (2004), 361–376.

[2] C. N. Campos, S. Dantasa and C. P. de Mello, Colouring clique-hypergraphs of circulant graphs, *Electron. Notes Discrete Math.*, **30** (2008), 189–194.

[3] D. Defossez, Clique-coloring some classes of odd-hole-free, *J. Graph Theory*, **53** (2006), 233–249.

[4] D. Duffus, B. Sands, N. Sauer and R. E. Woodrow, Two-coloring all two-element maximal antichains, *J. Comb. Theory Ser. A*, **57** (1991), 109–116.

[5] D. Duffus, H. A. Kierstead and W. T. Trotter, Fibres and ordered set coloring, *J. Comb. Theory Ser. A*, **58** (1991), 158–164.

[6] S. Gravier, C. T. Hoáng and F. Maffray, Coloring the hypergraph of maximal cliques of a graph with no long path, *Discrete Math.*, **272** (2003), 285–290.

[7] S. Gravier and R. Škrekovski, Coloring the clique hypergraph of graphs without forbidden structure, *Les cahiers du laboratoire Leibniz*, **83** (2003) (http://www-leibniz.imag.fr/LesCahiers/).

[8] J. Mycielski, Sur le coloriage des graphes, *Colloq. Math.*, **3** (1955), 161–162.

[9] S. Olariu, Paw-free graphs, *Information Processing Letters*, **28** (1988), 53–54.

[10] A. van Rooij and H. S. Wilf, The interchange graphs of a finte graph, *Acta Math. Acad. Sci. Hung.*, **16** (1965), 263–269.

[11] T. Wichianpaisarn and C. Uiyyasathian, Clique-chromatic numbers of line graphs, *Submitted.*

[12] D. B. West, *Introduction to graph theory*, Prentice Hall, New Jersey, (2001).