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M-C-PSEUDO INJECTIVE MODULES

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Abstract

In this paper we study M-C-pseudo injective module which is the generalization of pseudo injective module and we give an example of M-C-pseudo injective module which is not M-pseudo injective. We also study some properties related to co-Hopfian and Hopfian modules. We characterize the commutative semi-simple rings in terms of C-pseudo injective modules.

1 Introduction

Through out the paper rings are associative with identity and modules are unitary right R-modules. Let M and N be two R-modules. A module N is called (pseudo) M-injective, if for every submodule A of M any (monomorphism) homomorphism from A to N can be extended to a homomorphism from M to N. M is called (pseudo) quasi-injective, if it is (pseudo) M-injective. M and N are called relatively (pseudo) injective, if M is (pseudo)N-injective and N is (pseudo) M-injective. A ring R is said to be pseudo injective ring, if R_R is a pseudo injective module. A submodule K of a module M is said to be a closed submodule of M, if K has no proper essential extension inside M, i.e. whenever L is a submodule of M such that K is essential in L then $K = L$, equivalently a submodule H of M is called a complement of a submodule N of M, if H is maximal in the collection of submodules Q of M such that $Q \cap N = 0$, for detail see [6] and [8]. The idea of C-quasi injective was given by Tiwary et. al. in

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1979 [13]. A module N is called M-C-injective if for every closed submodule X of M, any homomorphism from X to N can be extended to a homomorphism from M to N. A module M is called C-quasi injective if it is M-C-injective. In this paper we generalize the idea of pseudo injective modules and C-quasi injective modules to C-pseudo injective modules. We give an example of M-C-pseudo injective modules which is not M-pseudo injective.
Consider the following condition for an R-module M :
(C_1) Every submodule of M is essential in a direct summand of M.
(C_2) If a submodule of M is isomorphic to a direct summand of M, then it is a direct summand of M itself.
(C_3) If A and B are direct summand of M with $A \cap B = 0$. Then $A \oplus B$ is also a direct summand of M.
A module M is called CS module (or extending module), if it satisfies (C_1) and it is called continuous (resp. quasi-continuous), if it satisfies (C_1) and (C_2) (resp.(C_1) and (C_3)). A module M is directly finite if and only if $\alpha\beta = 1$ implies that $\beta\alpha = 1$, $\forall\ \alpha, \beta \in \mathrm{End}(M)$, for detail see [10]. A module M is called co-Hopfian (Hopfian), if every injective (surjective) endomorphism of M is an isomorphism. Some properties of M-C-pseudo injective modules are studied and the concept of quasi-C-pseudo injective module is also introduced and we abbreviated it as C-pseudo injective module. We provide a characterization of commutative semi-simple rings in terms of C-pseudo injective modules. Finally, we give a sufficient condition for a C-pseudo injective module to be co-Hopfian.
Let M and N be two R-modules. A homomorphism $f : M \to N$ is said to be C-homomorphism, if $f(M)$ is closed submodule of N. A module M is called non-singular, if $Z(M) = 0$, where $Z(M) = \{m \in M : ann(m) \subseteq_e R_R\}$. For useful notation and terminology we refer to [1].

2 C-Pseudo Injective Modules

Definition 2.1. *A right R-module N is called M-C-pseudo injective if for every closed submodule K of M, any monomorphism from K to N can be extended to a homomorphism from M to N. If M is M-C-pseudo injective then it is called C-pseudo injective module.*

Now, we give an example of M-C-pseudo injective module which is not M-pseudo injective.

Example 2.2. *Let F be a field and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, $M_R = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, $N_R = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$, where M and N are right R-modules. Then, N is M-C-pseudo injective module but N is not M-pseudo injective.*

Proof:

Since $Q_R = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ is a right R-submodule of M_R.
Define $\phi : \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$ by $\phi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.
It is clear that ϕ is an isomorphism. For any $\beta : M_R \to N_R$ with
$\beta \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}$ for some $x \in F$. Then,
$\beta \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \beta \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$
$\beta \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} \cdot \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$
$\beta \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
So that $\beta = 0$. Hence, N is not M-pseudo injective module. It is clear that the submodule $\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ is not closed submodule of M. Thus, the only closed submodule of M are 0 and M itself. Hence N is M-C-pseudo injective module.

Example 2.3. *$\mathbb{Z}$ is C-pseudo injective module, but not $\mathbb{Z}$-injective module.*

Now, we discuss some properties of C-pseudo injective modules.

Proposition 2.4. *If N is M-C-pseudo injective module then any C-monomorphism $\alpha : N \to M$ splits.*

Proof: Let $\alpha : N \to M$ be C-monomorphism, *i*.e. $\alpha(N)$ is a closed submodule of M and $\alpha^{-1} : \alpha(N) \to N$ be inverse of α. As N is M-C-pseudo injective module then, there exists a homomorphism $\alpha^{'} : M \to N$ that extend α^{-1}. Set $u = \alpha^{'}\alpha$. Then, u is clearly an identity map on N. Hence, α splits.

Lemma 2.5. *Let $L \subseteq N \subseteq M$ be R-modules. If $L \subseteq_c N$, $N \subseteq_c M$ then $L \subseteq_c M$.*

Proof: For proof see [[5], 1.10].

Proposition 2.6. *If N is M-C-pseudo injective module then N is L-C-pseudo injective for any closed submodule L of M.*

Proof: Assume that X is a closed submodule of L, where L is a closed submodule of M then X is a closed submodule of M and $\alpha : X \to N$ is a monomorphism. As N is M-C-pseudo injective therefore α can be extended to a homomorphism $\alpha^* : M \to N$. The restriction $\alpha^*|_L$ is a homomorphism from L to N, which extends α. Hence, N is L-C-pseudo injective.

Proposition 2.7. *Every direct summand of C-pseudo injective module is C-pseudo injective module.*

Proof: Let M be C-pseudo injective module and N be a direct summand of M. Let L be a closed submodule of N, $i_1 : L \to N$ and $i_2 : N \to M$ be inclusions and let $\alpha : L \to N$ be a monomorphism. Since M is C-pseudo injective, therefore there exists $\beta : M \to M$ such that $\beta \circ \iota_2 \circ \iota_1 = \iota \circ \alpha \Rightarrow p \circ \beta \circ \iota_2 \circ \iota_1 = p \circ \iota \circ \alpha$, where $i : N \to M$, $p : M \to N$ are the inclusion and projection maps respectively. Take $\phi = p \circ \beta \circ \iota_2$ & $p \circ \iota = I_N$. Therefore, $\phi \circ \iota_1 = I \circ \alpha \Rightarrow \phi \circ \iota_1 = \alpha$.

Proposition 2.8. *Let M be C-pseudo injective module. Then every fully invariant closed submodule of M is C-pseudo injective module.*

Proof: Let N be a fully invariant closed submodule of M, let K be a closed submodule of N and let $\alpha : K \to N$ be a monomorphism. Since N is a closed submodule of M, it follows that K is also a closed submodule of M. Then there exists $\beta : M \to M$ that extends α. Note that $\beta(N) \subseteq N$, by hypothesis. Thus $\beta|_N : N \to N$ is a homomorphism. Hence, N is C-pseudo injective module.

Lemma 2.9. *Suppose that $L \subseteq K \subseteq M$ are R-modules. If $K \subseteq_c M$ then $K/L \subseteq_c M/L$.*

Proof: For proof see [[5], 1.10].

Lemma 2.10. *The submodule K is closed in M if and only if whenever Q is essential in M such that, $K \subset Q$, then Q/K is essential in M/K.*

Proof: For proof see [[5], 1.10].

Lemma 2.11. *If $K \subseteq_c M$, then the closed submodules of M/K are of the form H/K, where $H \subseteq_c M$ and $K \subseteq H$.*

Proof: Suppose $K \subseteq_c M$ and we prove that $H \subseteq_c M$. By lemma (2.9) above $H/K \subseteq_c M/K$ for every $H \subseteq_c M$ such that $K \subseteq H$. If $N \subseteq M$ is such that $H \subseteq_e N$, then by above lemma (2.10) $H/K \subseteq_e N/K$. Because $H/K \subseteq_c M/K$, we can conclude that $H = N$ and that $H \subseteq_c M$.

Proposition 2.12. *Let M_1 and M_2 be R-modules. If M_2 is M_1-C-injective module, then M_2 is M_1/N-C-pseudo injective for every closed submodule N of M_1.*

Proof: Let K/N be a closed submodule of M_1/N. Consider $\alpha : K/N \to M_2$ is a monomorphism and by lemma (2.11) above we have $K \subseteq_c M_1$. Let $\pi : M_1 \to M_1/N$ and $\pi' : K \to K/N$ be the canonical epimorphisms. As M_2 is M_1-C-injective, there exists $\beta : M_1 \to M_2$ that extends $\alpha\pi'$. Since $N \subseteq Ker\beta$, the existence of a homomorphism $\gamma : M_1/N \to M_2$ such that $\gamma\pi = \beta$ is garunteed. For every $a \in K$, $\gamma(a + N) = \gamma\pi(a) = \beta(a) = \alpha\pi'(a) = \alpha(a + N)$. Therefore γ extends α and M_2 is M_1/N-C- pseudo injective.

Proposition 2.13. *If $M_1 \oplus M_2$ is C-pseudo injective module, then M_1 and M_2 are mutually C-injective.*

Proof: Let $M_1 \oplus M_2$ be C-pseudo injective module. It is enough to show that M_1 is M_2-C-injective. Let K be a closed submodule of M_2 and $\phi : K \to M_1$ be a homomorphism. Define $\psi : K \to M_1 \oplus M_2$ by $\psi(a) = (\phi(\mathrm{a}),\mathrm{a})$, $\forall\ a \in K$. Then, ψ is a monomorphism. As $M_1 \oplus M_2$ is M_2-C-pseudo injective, therefore ψ can be extended to a homomorphism f from M_2 to $M_1 \oplus M_2$ i.e. $\psi = f \circ \iota$, where $\iota : K \to M_2$ is the inclusion map. Let $\pi_1 : M_1 \oplus M_2 \to M_1$ be the natural projection. Now, $\pi_1 \circ \psi = \pi_1 \circ f \circ \iota$, hence $\phi = \pi_1 \circ f \circ \iota$,Then, $\pi_1 \circ f$ is a homomorphism extending ϕ. Therefore, M_1 is M_2-C-injective.

Proposition 2.14. *If M is C-pseudo injective and N is a closed submodule of M, then any map $f : N \to M$ can be extended to M, provided that $Kerf \leq_e N$.*

Proof: Let M be C-pseudo injective module and $N \subseteq_c M$. Let $f : N \to M$ be given map with $Kerf \leq_e N$. Consider a map $g = (I_N - f) : N \to M$. Clearly, $Kerg = 0$ and hence g has an extension h to M because M is C-pseudo injective. Then, $I_M - h$ is an extension of f to M.

Proposition 2.15. *Let X, Y and M be R-modules with $X \cong Y$. If X is M-C-pseudo injective module, then Y is also M-C-pseudo injective module.*

Proof: Obvious.

Proposition 2.16. *Let M and N be two R-modules and $X = M \oplus N$. If M is N-C-pseudo injective module then for any complement submodule K in X of M with $K \cap N = 0$ and $\pi_N(K) \subseteq_c N$, where π_N is the projection from X onto N, then $M \oplus K = X$.*

Proof: Let K be a closed submodule in X of M with $K \cap N = 0$, $\pi_M : M \oplus K \to M$ and $\pi_N : M \oplus N \to N$ be the projections. As $\pi_N(K)$ is closed in N. Define $\theta : \pi_N(K) \to \pi_M(K)$ as follows : for $k \in K$ with $k = m + n$ $(m \in M, n \in N)$, $\theta(n) = m$. Then, θ is a monomorphism by $K \cap N = 0$, then $M \oplus K = X$ assumption. As M is N-C-pseudo injective therefore θ can be extended to some $g : N \to M$. Now, let us assume that $T = \{n + g(n) : n \in N\}$. Then clearly $M \oplus T = X$. Since K is closed $\Rightarrow T = K$. Hence, $M \oplus K = X$.

In [2] every extending module is C-injective module. Clearly, every C-injective module is C-pseudo injective module then it follows that every extending module is C-pseudo injective module, while the converse need not be true for example, the $\mathbb{Z}$ modules $M_1 = \mathbb{Q}$ and $M_2 = \mathbb{Z}/p\mathbb{Z}$ for a prime p, the $\mathbb{Z}$-module $M_1 \oplus M_2$ is C-pseudo injective but not extending [12].

In the next theorem, we provide a characterization of commutative semi-simple rings in terms of C-pseudo injective modules.

Theorem 2.17. *For a commutative ring R, the following conditions are equivalent:*
(1) *The direct sum of every two C-pseudo injective R-modules are C-pseudo injective modules;*
(2) *Every C-pseudo injective module is injective;*
(3) *R is semisimple artinian.*

Proof: (1) $\Rightarrow$ (2) Suppose that M is C-pseudo injective R-module and $E(M)$ is the injective hull of M. Then since $E(M)$ is injective, it is C-pseudo injective and by assumption $N = M \oplus E(M)$ is C-pseudo injective. Consider the injection maps $i_1 : M \to E(M)$, $i_2 : E(M) \to M \oplus E(M)$, $i_3 : M \to M \oplus E(M)$, the identity mapping $i : M \to M$, and the projection $p : M \oplus E(M) \to M$ so that $p \circ i_3 = i$. Now, $M \oplus E(M)$ is C-pseudo injective implies that there exists a homomorphism $g : M \oplus E(M) \to M \oplus E(M)$ such that $i_3 \circ i = g \circ i_2 \circ i_1 \Rightarrow p \circ i_3 \circ i = p \circ g \circ i_2 \circ i_1 \Rightarrow I_M = p \circ g \circ i_2 \circ i_1$ so that, $f = p \circ g \circ i_2$ therefore, $I_M = f \circ i_1 \Rightarrow M$ is isomorphic to a direct summand of $E(M)$ and hence injective.

(2) $\Rightarrow$ (3) Assume that every C-pseudo injective module is injective. Since every simple module is C-pseudo injective, it is injective and therefore R is a V-ring and therefore von-Neumann regular ring due to commutativity of R. Furthermore, every completely reducible R-module is C-pseudo injective, it is injective. By Kurshan [11] it follows that if the countable direct sum of injective hulls of simple modules is injective, then R is noetherian ring. Thus, R being noetherian and regular is semi-simple artinian.

(3) $\Rightarrow$ (1) R is semi-simple artinian this implies that every R-module is injective this implies that the direct sum of any two R-module is injective. Thus, every R-module is C-pseudo injective and the direct sum of two C-pseudo injective module is C-pseudo injective.

Remark 2.18. *In theorem (2.17) the commutativity of the ring is used to prove only* (2) $\Rightarrow$ (3).

Corollary 2.19. *[[13], Theorem(1.1)] For a commutative ring R, the following conditions are equivalent:*
(1) *The direct sum of every two C-injective R-modules is C-injective modules;*
(2) *Every C-injective module is injective;*
(3) *R is semi-simple artinian.*

Proposition 2.20. *Let M be C-pseudo injective and directly finite module. Then M is co-Hopfian.*

Proof: Let α be any one-one endomorphism of M and $I_M : M \to M$ be an identity map. Since M is C-pseudo injective, there exists $\beta : M \to M$ such that $\beta\alpha = I_M \Rightarrow \alpha\beta = I_M$ [10]. Which shows that α is onto. Hence, M is co-Hopfian.

Corollary 2.21. *Let M be C-injective and directly finite module. Then M is co-Hopfian.*

Proposition 2.22. *Let M be C-pseudo injective and Hopfian module. Then M is co-Hopfian.*

Proof: Since M is Hopfian, it follows that it is directly finite and hence, M is co-Hopfian.

Corollary 2.23. *Let M be C-injective and Hopfian module, then it is co-Hopfian.*

A submodule N of M is called fully invariant submodule of M if for every $f \in End(M)$, $f(N) \subseteq N$. If $M = K \oplus L$ and N is a fully invariant submodule of M, we have $N = N \cap K \oplus N \cap L$. A module M is called duo, if every submodule of M is fully invariant. A module M is said to have summand intersection property(SIP), if intersection of two summands of M is a direct summand of M.

Proposition 2.24. *A C-pseudo injective and duo module M has SIP.*

Proof: Suppose that $M = N \oplus N_1$ and $M = K \oplus K_1$. We show that $N \cap K$ is a direct summand of M. $N = N \cap M = N \cap (K \oplus K_1)$. Hence, $M = (N \cap K) \oplus (N \cap K_1) \oplus N_1$.

Lemma 2.25. *Let M be nonsingular right R-module, and $N_i \subseteq_c M$ ($i \in I$), then $\cap_i N_i \subseteq_c M$.*

Proof: For proof see [[8], Proposition (7.44)].

Lemma 2.26. *[[13], Lemma (1.1)] If a closed submodule C of a quasi-injective module M, is isomorphic to a submodule C', then C' is a closed submodule of M.*

Definition 2.27. *An R-module M is called an icp-injective module, if for every monomorphism from an isomorphic copy (in M) of a closed submodule of M into M, can be extended to an endomorphism of M.*

Lemma 2.28. *Direct summand of an icp-injective module is an icp-injective module.*

Proof: Proof is same as Proposition(2.7).

Lemma 2.29. *A C-pseudo injective module M in which the isomorphic copy (in M) of a closed submodule is closed, is icp-injective.*

Proof: Let C be an isomorphic copy of closed submodule of M. Then C is a closed submodule in M. Hence, every monomorphism from C into M can be extended to an endomorphism of M by C-pseudo injectivity of M.

Proposition 2.30. *If R is a pseudo injective ring, then every non-zero divisors of R is invertible in R. Consequently, R coincide with the quotient ring of R.*

Proof: Let c be a non-zero divisor of R. Define $f : cR \to R$ by $f(ca) = a\ \forall a \in R$. Then f is well defined monomorphism. Now, $i : cR \to R$ is the inclusion map, since R is pseudo injective ring, then there exists a homomorphism $h : R \to R$ such that $h \circ i = f$. If $h(1) = u$ for some $u \in R$, then $1 = f(c) = h \circ i(c) = h(c) = h(1)c = uc$ this implies that $c = cuc \Rightarrow (1-cu)c = 0 \Rightarrow cu = 1$. Hence, c is invertible.

Proposition 2.31. *Let M be a nonsingular R-module and $S = End(M)$ and $I = \{f \in S : Kerf \subseteq_c M\}$, then I is a two sided ideal of $S = End(M_R)$.*

Proof: Let $f, g \in I \Longrightarrow Kerf \subseteq_c M$ and $Kerg \subseteq_c M$. Then $Kerf \cap Kerg \subseteq_c M$ as M is nonsingular (by lemma(2.25)). Since $Kerf \cap Kerg \subseteq Ker(f-g) \subseteq M$ therefore $Kerf \cap Kerg \subseteq_c Ker(f-g)$. Claim that $Ker(f-g) \subseteq_c M$. For this, let us assume that Ker$(f-g)$ is not closed in M, then there exists a non zero proper essential extension L of $Ker(f-g)$ in M, *i.*e. $Ker(f-g) \subseteq_e L$. Let $x \in L - Ker(f-g)$ and $K =< x >$ be a submodule of L. Let $z \in Ker(f-g) \cap K \Rightarrow (f-g)(z) = 0$ and $z = xr \in K$ for some non zero r and x in K. $(f-g)(xr) = 0 \Rightarrow r(f-g)(x) = 0 \Rightarrow x \in Ker(f-g)$ which is absurd by assumption. So, $L = Ker(f-g)$ hence $Ker(f-g) \subseteq_c M \Rightarrow f - g \in I$. Next, let $h \in S$ and $f \in I$, $Kerf \subseteq_c M$. Now, $Kerfh \subseteq M$, $Kerf \subseteq_c Kerfh \subseteq M$ by the previous argument $Kerfh \subseteq_c M \Rightarrow fh \in I$. Similarly, $hf \in I$. Hence, I is a two sided ideal of S.

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