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A UNIFIED CONTINUED FRACTION EXPANSION AND INDEPENDENCE CHARACTERIZATION

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Abstract

An algorithm to construct a general continued fraction expansion for elements in a discrete-valued non-archimedean fields $(K, |\cdot|)$ is devised. Such continued fraction takes the form

$$\frac{b_1}{a_1+}\frac{b_2}{a_2+}\cdots\frac{b_n}{a_n+}\cdots,$$

where the elements a_n, b_n are subject to the condition $|a_n| > |b_n|$. Several examples are given to show that this algorithm yields almost all known continued fraction expansions as special cases. Criteria for algebraic and linear dependences of certain classes of such continued fractions are derived.

1 Introduction

A continued fraction expansion is an expression of the form

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{\ddots\, a_{n-1} + \cfrac{b_n}{\ddots}}}} := a_0 + \frac{b_1}{a_1+}\frac{b_2}{a_2+}\cdots\frac{b_n}{a_n+}\cdots,$$

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where the elements $b_1, b_2, b_3, \ldots$ are called its *partial numerators*, and $a_1, a_2, a_3, \ldots$ its *partial denominators*. When all $b_i = 1$ $(i \geq 1)$, it is usually referred to as a *regular or simple continued fraction expansion*. In 2002, Y. Hartono, C. Kraaikamp and F. Schweiger [3] introduced a new continued fraction expansion, called Engel continued fraction (or ECF) expansion, of the real numbers in the interval $(0,1)$. The ECF map $T_E : [0,1) \to [0,1)$ is given by

$$T_E(x) := \frac{1}{\lfloor 1/x \rfloor}\left(\frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor\right), \; x \neq 0; \quad T_E(0) := 0.$$

For each $x \in (0,1)$, the ECF map generates a continued fraction expansion of x of the form

$$x = \frac{1}{a_1+}\,\frac{a_1}{a_2+}\cdots\frac{a_{n-1}}{a_n+}\cdots, \quad a_n = a_n(x) := \left\lfloor 1/T_E^{n-1}(x) \right\rfloor \; (n \geq 1),$$

where the digits a_n satisfy the condition $1 \leq a_n \leq a_{n+1}$. Motivated by Hartono-Kraaikamp-Schweiger's work, we devise an algorithm that enables us to uniquely construct a continued fraction expansion, henceforth called a JR-continued fraction, for each element in a field completed with respect to a discrete non-archimedean valuation. Several well-known examples are then shown to be special cases of JR-continued fractions. In the last section, general criteria for algebraic and linear independences are derived for JR-continued fractions.

2 Algorithm

Let us start by briefly recalling some known facts, for a general reference, see [5]. Let K be a field completed with respect to a discrete non-archimedean valuation $|\cdot|$ and let

$$\mathcal{O} := \{\alpha \in K \; ; \; |\alpha| \leq 1\}$$

be its ring of integers. The set

$$\mathcal{P} := \{\alpha \in K \; ; \; |\alpha| < 1\}$$

is an ideal in $\mathcal{O}$, which is both a maximal ideal and a principal ideal, generated by a prime element $\tau \in K$. The quotient ring $\mathcal{O}/\mathcal{P}$ is a field, called the residue class field. Let $\mathcal{A} \subset \mathcal{O}$ be a set of representatives of $\mathcal{O}/\mathcal{P}$. Each $\alpha \in K \setminus \{0\}$ is uniquely representable as

$$\alpha = \sum_{n=N}^{\infty} c_n \tau^n \qquad (N \in \mathbb{Z}, \; c_n \in \mathcal{A}, \; c_N \neq 0);$$

such representation is usually referred to as its *canonical representation*. The non-archimedean valuation of α is so normalized that $|\alpha| = e^{-N}$, with $|0| := 0$.

The *head part* $\langle\alpha\rangle$ of α is defined as the finite series

$$\langle\alpha\rangle = \sum_{n=N}^{0} c_n \tau^n \quad \text{if } N \le 0, \text{ and 0 otherwise.}$$

Denote the set of all head parts by

$$S := \{\langle\alpha\rangle \ ; \ \alpha \in K\}.$$

We are now ready to introduce our continued fraction algorithm.

Let $\{b_i\}_{i\ge 1}$ be a sequence in $K\setminus\{0\}$, each of whose elements b_i is either fixed or is uniquely determined from α and previously known parameters b_j, a_j $(j < i)$ arising from the algorithm.

For convenience, we consider $\alpha \in K$ such that $|\alpha| < 1$. Let $A_1 := \alpha \ne 0$. Assume that $b_1 \in K \setminus \{0\}$ is subject to the condition that

$$|b_1/A_1| \ge 1. \tag{1}$$

Define $a_1 = \left\langle \frac{b_1}{A_1} \right\rangle \in S \setminus \{0\}$.

- If $a_1 = b_1/A_1$, then the process stops and we write
$$\alpha = A_1 = \frac{b_1}{a_1}.$$
- When $a_1 \ne b_1/A_1$, we have $0 < |b_1/A_1 - a_1| < 1$. Assume that $b_2 \in K \setminus \{0\}$ is subject to the condition that the element $A_2 = \frac{1}{b_2}\left(\frac{b_1}{A_1} - a_1\right) \ne 0$ satisfies
$$0 < |A_2| < 1. \tag{2}$$
Thus,
$$\alpha = A_1 = \frac{b_1}{a_1 + b_2 A_2}.$$

Next, define $a_2 = \langle 1/A_2 \rangle \in S \setminus \{0\}$.

- If $a_2 = 1/A_2$, then the process stops and we write
$$\alpha = \frac{b_1}{a_1 + b_2 A_2} = \frac{b_1}{a_1 + \dfrac{b_2}{a_2}} = \frac{b_1}{a_1 +}\ \frac{b_2}{a_2}.$$
- When $a_2 \ne 1/A_2$, we have $0 < |1/A_2 - a_2| < 1$. Assume that $b_3 \in K\setminus\{0\}$ is subject to the condition that the element $A_3 = \frac{1}{b_3}\left(\frac{1}{A_2} - a_2\right) \ne 0$ satisfies
$$0 < |A_3| < 1. \tag{3}$$

Thus,

$$\alpha = \frac{b_1}{a_1 + b_2 A_2} = \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + b_3 A_3}} = \frac{b_1}{a_1+}\ \frac{b_2}{a_2 + b_3 A_3}.$$

Continuing this process, if $A_i \neq 0 \ \ (i \geq 2)$ has already been constructed with $0 < |A_i| < 1$, then define $a_i = \langle 1/A_i \rangle \in S \setminus \{0\}$.

- If $a_i = 1/A_i$, then the process stops and we have a finite continued fraction expansion
$$\alpha = \frac{b_1}{a_1+}\ \frac{b_2}{a_2+} \cdots \frac{b_i}{a_i}.$$
- When $a_i \neq 1/A_i$, we have $0 < |1/A_i - a_i| < 1$. Assume that $b_{i+1} \in K \setminus \{0\}$ is subject to the condition that the element $A_{i+1} = \frac{1}{b_{i+1}}\left(\frac{1}{A_i} - a_i\right) \neq 0$ satisfies
$$0 < |A_{i+1}| < 1, \tag{4}$$
and so
$$\alpha = \frac{b_1}{a_1+}\ \frac{b_2}{a_2+} \cdots \frac{b_i}{a_i + b_{i+1} A_{i+1}}.$$

Observe that $|a_1| = |b_1|/|A_1| > |b_1|$, since $0 < |b_2 A_2| = |b_1/A_1 - a_1| < 1$ and $0 < |b_{i+1} A_{i+1}| = |1/A_i - a_i| < 1 \ (i \geq 2)$, we have

$$|a_{i+1}| = 1/|A_{i+1}| > |b_{i+1}| \qquad (i \geq 1). \tag{5}$$

Note that if the b_i's belong to $S \setminus \{0\}$, then the requirements (1), (2), (3) and (4) hold automatically.

Summing up, we see that the algorithm yields a JR-continued fraction expansion of the form

$$\alpha = \frac{b_1}{a_1+}\ \frac{b_2}{a_2+} \cdots \frac{b_n}{a_n + b_{n+1} A_{n+1}}$$

where $a_i \in S \setminus \{0\}$ and b_i are subject to (5). If $a_1 = b_1/A_1$ or $a_n = 1/A_n \ \ (n \geq 2)$, then

$$\alpha = \frac{b_1}{a_1+}\ \frac{b_2}{a_2+} \cdots \frac{b_n}{a_n},$$

i.e., the JR-continued fraction expansion of α is finite. If $a_1 \neq b_1/A_1$ and $a_n \neq 1/A_n \ \ (n \geq 2)$, we now proceed to show that this JR-continued fraction expansion converges.

Define two sequences (C_n), (D_n) as follows:

$$C_{-1} = 1, \quad C_0 = 0, \quad C_{n+1} = a_{n+1} C_n + b_{n+1} C_{n-1} \ \ (n \geq 0)$$
$$D_{-1} = 0, \quad D_0 = 1, \quad D_{n+1} = a_{n+1} D_n + b_{n+1} D_{n-1} \ \ (n \geq 0).$$

The following proposition is easily established by induction.

Proposition 1. *For any* $n \geq 0,\ \beta \in K \setminus \{0\}$, *we have*

(i) $\dfrac{\beta C_n + b_{n+1} C_{n-1}}{\beta D_n + b_{n+1} D_{n-1}} = \dfrac{b_1}{a_1+}\ \dfrac{b_2}{a_2+} \cdots \dfrac{b_{n+1}}{\beta}$

(ii) $\dfrac{C_n}{D_n} = \dfrac{b_1}{a_1+}\ \dfrac{b_2}{a_2+} \cdots \dfrac{b_n}{a_n} \qquad (n \geq 1)$

(iii) $C_n D_{n-1} - C_{n-1} D_n = (-1)^{n-1} b_1 b_2 \cdots b_n \quad (n \geq 1)$

(iv) $|C_1| = |b_1|,\ |C_n| = |b_1 a_2 a_3 \cdots a_n| \qquad (n \geq 2)$

(v) $|D_n| = |a_1 a_2 \cdots a_n| \neq 0 \qquad (n \geq 1).$

From Proposition 1, we have

$$\frac{C_n}{D_n} = \frac{a_n C_{n-1} + b_n C_{n-2}}{a_n D_{n-1} + b_n D_{n-2}} = \frac{b_1}{a_1+}\ \frac{b_2}{a_2+} \cdots \frac{b_n}{a_n} \qquad (n \geq 1),$$

and so C_n/D_n is called the n^{th} convergent of the JR-continued fraction expansion of α. From the algorithm and Proposition 1 (i), we obtain

$$\alpha = \frac{b_1}{a_1+}\ \frac{b_2}{a_2+} \cdots \frac{b_n}{a_n + b_{n+1} A_{n+1}} = \frac{(a_n + b_{n+1} A_{n+1})\, C_{n-1} + b_n C_{n-2}}{(a_n + b_{n+1} A_{n+1})\, D_{n-1} + b_n D_{n-2}}.$$

Using Proposition 1(ii) and (iii), it is easy to check that

$$\alpha - \frac{C_n}{D_n} = \frac{(-1)^n b_1 b_2 \cdots b_n b_{n+1} A_{n+1}}{D_n \left((a_n + b_{n+1} A_{n+1})\, D_{n-1} + b_n D_{n-2} \right)}.$$

From

$$|a_n| \geq 1 > |b_{n+1}|/|a_{n+1}| = |b_{n+1} A_{n+1}|,$$

we get $|a_n + b_{n+1} A_{n+1}| = |a_n|$, and so $|\,(a_n + b_{n+1} A_{n+1})\, D_{n-1} + b_n D_{n-2}| = |a_n D_{n-1}|$. Thus,

$$\left| A_1 - \frac{C_n}{D_n} \right| = \frac{|b_1 b_2 \cdots b_{n+1}|}{|D_n||D_{n+1}|} \to 0 \qquad (n \to \infty),$$

showing that C_n/D_n converges to α, which enables us to write

$$\alpha = \frac{b_1}{a_1+}\ \frac{b_2}{a_2+}\ \cdots \frac{b_n}{a_n+} \cdots .$$

To show uniqueness, suppose that $\alpha \in K \setminus \{0\},\ |\alpha| < 1$, has two such JR-continued fraction expansions

$$\frac{b_1}{a_1+}\ \frac{b_2}{a_2+}\ \cdots \frac{b_n}{a_n+} \cdots = \alpha = \frac{b_1'}{a_1'+}\ \frac{b_2'}{a_2'+}\ \cdots \frac{b_n'}{a_n'+} \cdots ,$$

where $a_i, a_i' \in S \setminus \{0\}$ and the b_i, b_i' are subject to the same requirements as elaborated above. Observe that we have

$$\left| \frac{b_i}{a_i+} \, \frac{b_{i+1}}{a_{i+1}+} \cdots \right| \le \frac{|b_i|}{|a_i|} < 1 \qquad (i \ge 1) \tag{6}$$

with the same relations for b_i', a_i' $(i \ge 1)$. From the construction requirement, we have $b_1 = b_1'$ which implies that

$$a_1 + \frac{b_2}{a_2+} \, \frac{b_3}{a_3+} \, \cdots = a_1^{'} + \frac{b_2^{'}}{a_2^{'}+} \, \frac{b_3^{'}}{a_3^{'}+} \, \cdots .$$

Since $a_1, a_1^{'} \in S$, using (6), we get

$$a_1 = a_1^{'} \;\; \text{and} \;\; \frac{b_2}{a_2+} \, \frac{b_3}{a_3+} \, \frac{b_4}{a_4+} \cdots = \frac{b_2^{'}}{a_2^{'}+} \, \frac{b_3^{'}}{a_3^{'}+} \, \frac{b_4^{'}}{a_4^{'}+} \cdots .$$

Since $a_1 = a_1^{'}$, from the definition, we have $b_2 = b_2^{'}$. Continuing in the same manner, we get $a_i = a_i^{'}$, $b_i = b_i^{'}$ for all i. The following theorem summarizes our results so far obtained.

Theorem 1. *Each $\alpha \in K \setminus \{0\}$ with $|\alpha| < 1$, can be represented uniquely by a JR-continued fraction expansion of the form*

$$\alpha = \frac{b_1}{a_1+} \, \frac{b_2}{a_2+} \, \cdots \frac{b_n}{a_n+} \cdots ,$$

where $a_i \in S \setminus \{0\}$ and the sequence $\{b_i\}_{i=1}^{\infty} \subseteq K \setminus \{0\}$ is either fixed or is uniquely determined from α and previously known parameters b_j, a_j $(j < i)$. Moreover, the partial numerators and denomintors are subject to the condition, which will henceforth be referred to as the **ab-condition**,

$$|a_i| > |b_i| \qquad (i \ge 1). \tag{7}$$

3 Examples

We turn now to specific examples.

Example 1. Let K be a field completed with respect to a discrete non-archimedean valuation $|\cdot|$. Taking all $b_i = 1$ $(i \ge 1)$ in Theorem 1, we deduce that every $\alpha \in K \setminus \{0\}$, $|\alpha| < 1$, has a unique regular continued fraction expansion of the form

$$\alpha = \frac{1}{a_1+} \, \frac{1}{a_2+} \cdots \frac{1}{a_n+} \cdots ,$$

where $a_i \in S \setminus \{0\}$ are subject to the ab-condition, i.e., $|a_i| > 1 \quad (i \geq 1)$. This is the well-known classical regular continued fraction.

Example 2. Let K be a field completed with respect to a discrete non-archimedean valuation $|\cdot|$. Taking $b_1 = 1,\ b_{i+1} = a_i\ (i \geq 1)$ in Theorem 1, we deduce that every $\alpha \in K \setminus \{0\},\ |\alpha| < 1$, has a unique continued fraction expansion of the form

$$\alpha = \frac{1}{a_1+}\,\frac{a_1}{a_2+}\cdots\frac{a_{n-1}}{a_n+}\cdots,$$

where $a_i \in S\setminus\{0\}\ (i \geq 1)$ are subject to the ab-condition, i.e., $|a_{i+1}| > |b_{i+1}| = |a_i| \quad (i \geq 1)$. This continued fraction may be regarded as a non-archimedean analogue of the real ECF expansion due to Hartono-Kraaikamp-Schweiger, [3].

Example 3. Let K be a field completed with respect to a discrete non-archimedean valuation $|\cdot|$. Taking $b_1 = 1,\ b_{i+1} = a_i^2 - a_i + 1\ (i \geq 1)$ in Theorem 1, we deduce that every $\alpha \in K \setminus \{0\},\ |\alpha| < 1$, has a unique continued fraction expansion of the form

$$\alpha = \frac{1}{a_1+}\,\frac{a_1^2 - a_1 + 1}{a_2+}\cdots\frac{a_{n-1}^2 - a_{n-1} + 1}{a_n+}\cdots,$$

where $a_i \in S \setminus \{0\}\ (i \geq 1)$ are subject to the ab-condition, i.e., $|a_{i+1}| > |b_{i+1}| = |a_i^2 - a_i + 1| \quad (i \geq 1)$. This continued fraction may be regarded as a non-archimedean analogue of the real Sylvester continued fraction expansion due to A. H. Fan, B. W. Wang and J. Wu, [2].

Example 4. Let $K = \mathbb{Q}_p$ be the field of p-adic numbers, i.e., the completion of $\mathbb{Q}$ with respect to the $p-$adic valuation, $|\cdot|_p$, so normalized that $|p|_p = p^{-1}$. Here, the ring of $p-$adic integers is $\mathcal{O} = \mathbb{Z}_p$. Each $\alpha \in p\mathbb{Z}_p \setminus \{0\}$ is uniquely representable in the form

$$\alpha = \sum_{n=N}^{\infty} c_n p^n \qquad (N \in \mathbb{N},\ c_n \in \{0, 1, \ldots, p-1\},\ c_N \neq 0).$$

There are two well-known p-adic continued fraction expansions, due respectively to Ruban ([6]) and Schneider ([7]).

4A. The p–adic Ruban continued fraction ([6]) of $\alpha \in p\mathbb{Z}_p \setminus \{0\}$ is of the form

$$\alpha = \frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots,$$

where the a_i's are of the form

$$c_{-m}p^{-m} + c_{-m+1}p^{-m+1} + \cdots + c_0 \quad (m \in \mathbb{N}),\ \ c_j \in \{0, 1, \ldots, p-1\},\ c_{-m} \neq 0.$$

This is a JR-continued fraction with all $b_i = 1$. The ab-condition (7) holds trivially.

4B. The p–adic Schneider continued fraction ([7]) of $\alpha \in p\mathbb{Z}_p \setminus \{0\}$ is of the form

$$\alpha = \frac{b_1}{a_1+}\, \frac{b_2}{a_2+}\, \frac{b_3}{a_3+} \cdots ,$$

where $a_i \in \{0, 1, \ldots, p-1\}$, $b_1 = |\alpha|_p^{-1}$, and each b_i is of the form p^s $(s \in \mathbb{N})$ and is uniquely determined form α and previously known a_j, b_j $(j < i)$.

Example 5. Let $\mathbb{F}$ be a field and let

$$\mathbb{F}\left(\left(x^{-1}\right)\right) := \left\{\frac{c_r}{x^r} + \frac{c_{r+1}}{x^{r+1}} + \frac{c_{r+2}}{x^{r+2}} + \cdots ;\ r \in \mathbb{Z},\ c_i \in \mathbb{F},\ c_r \neq 0\right\}$$

be the completion of the rational function field $\mathbb{F}(x)$ with respect to the non-archimedean degree valuation, $|\cdot|_\infty$, so normalized that $|x^{-1}|_\infty = e^{-1}$. Let $\{b_i\}_{i=1}^{\infty}$ be a fixed sequence in $\mathbb{F}[x] \setminus \{0\}$. By Theorem 1, each $\alpha \in \mathbb{F}\left(\left(x^{-1}\right)\right) \setminus \{0\}$, $|\alpha|_\infty < 1$, has a unique JR-continued fraction expansion of the form

$$\alpha = \frac{b_1}{a_1+}\, \frac{b_2}{a_2+}\, \cdots \frac{b_n}{a_n+} \cdots ,$$

where $a_i \in \mathbb{F}[x] \setminus \{0\}$ are subject to the ab-condition, i.e., $|a_i|_\infty > |b_i|_\infty$ $(i \geq 1)$. The JR-continued fraction expansion in this case is indeed the non-regular continued fraction expansion constructed in [4].

Example 6. Let $\mathbb{F}$ be a field and let π be a prime element in $\mathbb{F}[x]$. The field

$$\mathbb{F}((\pi)) := \left\{c_r\pi^r + c_{r+1}\pi^{r+1} + c_{r+2}\pi^{r+2} + \cdots ;\ r \in \mathbb{Z},\ c_i \in \mathbb{F}[x],\ \deg c_i < \deg \pi,\ c_r \neq 0\right\}$$

of all formal Laurent series in π is the completion of $\mathbb{F}[x]$ with respect to the π-adic valuation, $|\cdot|_\pi$, so normalized that $|\pi|_\pi = e^{-\deg \pi}$. Its ring of integers is the set of formal power series

$$\mathbb{F}[[\pi]] := \left\{c_0 + c_1\pi + c_2\pi^2 + \cdots ;\ \ c_i \in \mathbb{F}[x],\ \deg c_i < \deg \pi\right\},$$

and the set of head parts is

$$S := \left\{c_r\pi^r + \cdots + c_{-1}\pi^{-1} + c_0;\ r \leq 0,\ c_i \in \mathbb{F}[x],\ \deg c_i < \deg \pi\right\}.$$

By Theorem 1, each $\alpha \in \pi\mathbb{F}[[\pi]] \setminus \{0\}$ is uniquely represented as a JR-continued fraction of the form

$$\alpha = \frac{b_1}{a_1+}\, \frac{b_2}{a_2+} \cdots \frac{b_n}{a_n+} \cdots , \tag{8}$$

where $a_i \in S \setminus \{0\}$ and b_i are subject to the ab-condition. There are various particular examples of JR-continued fractions in this setting. Let us mention two specific ones.

6A. The π-adic Ruban continued fraction is constructed in exactly the same manner as the p-adic Ruban continued fraction mentioned in Example 4A, i.e., each $\alpha \in \pi\mathbb{F}[[\pi]] \setminus \{0\}$ is uniquely representable as

$$\alpha = \frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots,$$

where the a_i's are of the form

$$c_{-m}\pi^{-m}+c_{-m+1}\pi^{-m+1}+\cdots+c_0 \ \ (m \in \mathbb{N}),\ \ c_j \in \mathbb{F}[x],\ \deg c_j < \deg \pi,\ c_{-m} \neq 0.$$

This is a JR-continued fraction with all $b_i = 1$. The ab-condition (7) holds trivially.

6B. The π-adic Schneider continued fraction is constructed in exactly the same manner as the p-adic Schneider continued fraction mentioned in Example 4B, i.e., each $\alpha \in \pi\mathbb{F}[[\pi]] \setminus \{0\}$ is uniquely representable as

$$\alpha = \frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\,\frac{b_3}{a_3+}\cdots,$$

where $a_i \in \mathbb{F}[x] \setminus \{0\}$, $\deg a_i < \deg \pi$, each b_i is of the form π^s $(s \in \mathbb{N})$ and is uniquely determined form α and previously known a_j, b_j $(j < i)$.

4 Independence

In this section, criteria for algebraic and/or linear independences of elements in $\mathbb{F}((\pi))$, as expounded in Example 6, are established along the same line as those in [1]. We begin with algebraic independence.

Theorem 2. *Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}((\pi)) \setminus \{0\}$. Assume that there are polynomials $C_{N,j}, D_{N,j}(\neq 0) \in \mathbb{F}[x]$ $(N \in \mathbb{N},\ 1 \le j \le n)$ such that*

$$D_{N,j}\alpha_j \neq C_{N,j}, \quad M_{N,j} := \max\{|C_{N,j}|_\infty, |D_{N,j}|_\infty\} \to \infty \quad (N \to \infty),$$

and

$$\lim_{N\to\infty} \frac{|\alpha_{j-1} - C_{N,j-1}/D_{N,j-1}|_\pi}{|\alpha_j - C_{N,j}/D_{N,j}|_\pi} = 0 \qquad (j = 2, \ldots, n) \tag{9}$$

provided $n \ge 2$. Assume further that for each positive real number E, there is an $N_0 = N_0(M) \in \mathbb{N}$ such that

$$\left|\alpha_j - \frac{C_{N,j}}{D_{N,j}}\right|_\pi \le \frac{1}{(M_{N,1}M_{N,2}\cdots M_{N,j})^E} \qquad (N \ge N_0\ ;\ j = 1, 2, \ldots, n). \tag{10}$$

Then $\alpha_1, \alpha_2, \ldots, \alpha_n$ are algebraically independent over $\mathbb{F}(x)$.

Proof. We proceed by induction on n. For $n = 1$, suppose that α_1 is algebraic of degree $m \geq 1$ over $\mathbb{F}(x)$. If $m = 1$, then $\alpha_1 = P/Q$ for some $P, Q \in \mathbb{F}[x] \setminus \{0\}$. For all $N \in \mathbb{N}$, we get

$$0 \neq |D_{N,1}\alpha_1 - C_{N,1}|_\pi = \frac{|D_{N,1}P - C_{N,1}Q|_\pi}{|Q|_\pi} \geq \frac{1}{|D_{N,1}P - C_{N,1}Q|_\infty \, |Q|_\pi},$$

and

$$|D_{N,1}P - C_{N,1}Q|_\infty \leq \max\{|D_{N,1}P|_\infty, |C_{N,1}Q|_\infty\} \leq M_{N,1}K,$$

$K := \max\{|P|_\infty, |Q|_\infty\}$, which by the product formula implies that

$$|D_{N,1}\alpha_1 - C_{N,1}|_\pi \geq \frac{1}{M_{N,1}K|Q|_\pi} = \frac{K_1}{M_{N,1}}, \quad K_1 := 1/K|Q|_\pi.$$

By (10), there is an $N_1 = N_1(2)$ such that, for all $N \geq N_1$,

$$\frac{K_1}{M_{N,1}} \leq |D_{N,1}\alpha_1 - C_{N,1}|_\pi = |D_{N,1}|_\pi \left|\alpha_1 - \frac{C_{N,1}}{D_{N,1}}\right|_\pi \leq \frac{1}{M_{N,1}^2},$$

which is a contradiction. For $m > 1$, by Uchiyama's Theorem ([8]), for $F((\pi))$ there is a constant $K_2 > 0$ such that

$$|D_{N,1}\alpha_1 - C_{N,1}|_\pi \geq \frac{K_2}{M_{N,1}^m} \qquad (N \in \mathbb{N}).$$

By (10), there is an $N_2 = N_2(m+1)$ such that, for all $N \geq N_2$,

$$\frac{K_2}{M_{N,1}^m} \leq |D_{N,1}\alpha_1 - C_{N,1}|_\pi \leq \frac{|D_{N,1}|_\pi}{M_{N,1}^{m+1}} \leq \frac{1}{M_{N,1}^{m+1}},$$

which is a contradiction. Thus, α_1 is transcendental, and we are done in the case $n = 1$.

Now consider $n > 1$. Assume the assertion of the theorem holds up to $n-1$, but is false for n. Then there would exist a polynomial $f(T_1, T_2, \ldots, T_n) \in \mathbb{F}[x][T_1, \ldots, T_n] \setminus \{0\}$ of minimal total degree such that $f(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0$. Expanding f about $(\alpha_1, \ldots, \alpha_n)$, we get

$$f(T_1, T_2, \ldots, T_n) = \sum h_{(\nu)} (T_1 - \alpha_1)^{\nu_1} \cdots (T_n - \alpha_n)^{\nu_n},$$

where $(\nu) = (\nu_1, \nu_2, \ldots, \nu_n)$, and

$$h_{(\nu)} := h_{(\nu_1,\nu_2,\ldots,\nu_n)} = \frac{1}{(\nu_1 + \nu_2 + \cdots + \nu_n)!} \frac{\partial^{\nu_1+\nu_2+\cdots+\nu_n} f(\alpha_1, \alpha_2, \ldots, \alpha_n)}{\partial T_1^{\nu_1} \partial T_2^{\nu_2} \cdots \partial T_n^{\nu_n}}.$$

Clearly,

$$h_{(0,\ldots,0)} = f(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0.$$

Set

$$\mathcal{H}_n\left(T_1, T_2, \ldots, T_n\right) := \frac{\partial}{\partial T_n} f\left(T_1, T_2, \ldots, T_n\right),$$
$$H_i = h_{(0,\ldots,0,1,0,\ldots,0)} \qquad (i = 1, 2, \ldots, n),$$

where the digit 1 is at the i^{th} position. Observe that T_n occurs in f. Thus, $\mathcal{H}_n\left(T_1, T_2, \ldots, T_n\right) \not\equiv 0$ and $H_n = \mathcal{H}_n\left(\alpha_1, \alpha_2, \ldots, \alpha_n\right)$.

Next, we show that $H_n \neq 0$. Suppose not. If T_n occurs in $\mathcal{H}_n\left(T_1, \ldots, T_n\right)$, then $(\alpha_1, \ldots, \alpha_n)$ is a root of a nonzero polynomial whose degree is lower than that of f, which is a contradiction. Thus, T_n does not occur in $\mathcal{H}_n\left(T_1, \ldots, T_n\right)$. This means that $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are algebraically dependent, contradicting the induction hypothesis. Thus, $H_n \neq 0$.

Let

$$\delta_j(N) = \frac{C_{N,j}}{D_{N,j}} - \alpha_j \qquad (j = 1, 2, \ldots, n).$$

Since $D_{N,j}\alpha_j \neq C_{N,j}$, we get $|\delta_n(N)|_\pi \neq 0$. Now

$$\begin{aligned} f\left(\frac{C_{N,1}}{D_{N,1}}, \frac{C_{N,2}}{D_{N,2}}, \ldots, \frac{C_{N,n}}{D_{N,n}}\right) &= \sum_{(\nu)} h_{(\nu)} \delta_1(N)^{\nu_1} \cdots \delta_n(N)^{\nu_n} \\ &= \sum_{i=1} H_i \delta_1(N) + \sum_{\nu_1+\cdots+\nu_n \geq 2} h_{(\nu)} \delta_1(N)^{\nu_1} \cdots \delta_n(N)^{\nu_n} \\ &= \delta_n(N)\left(\left(H_1 \frac{\delta_1(N)}{\delta_n(N)} + \cdots + H_{n-1}\frac{\delta_{n-1}(N)}{\delta_n(N)} + H_n\right) + O\left(|\delta_n(N)|_\pi\right)\right). \end{aligned}$$

By hypotheses (9) and (10), we see that

$$\begin{aligned} &\left| H_1 \frac{\delta_1(N)}{\delta_n(N)} + \cdots + H_{n-1}\frac{\delta_{n-1}(N)}{\delta_n(N)} + O\left(|\delta_n(N)|_\pi\right)\right|_\pi \\ &\quad \leq \max\left\{ \left|H_1 \frac{\delta_1(N)}{\delta_n(N)}\right|_\pi, \ldots, \left|H_{n-1}\frac{\delta_{n-1}(N)}{\delta_n(N)}\right|_\pi, O\left(|\delta_n(N)|_\pi\right)\right\} \to 0 \quad (N \to \infty), \end{aligned}$$

which yields, when N is large enough,

$$\left| f\left(\frac{C_{N,1}}{D_{N,1}}, \frac{C_{N,2}}{D_{N,2}}, \ldots, \frac{C_{N,n}}{D_{N,n}}\right)\right|_\pi = |\delta_n(N) H_n|_\pi \neq 0.$$

Let $m_1, m_2, \ldots, m_n$ be the degrees of f in $T_1, T_2, \ldots, T_n$, respectively. Then

$$D_{N,1}^{m_1} \cdots D_{N,n}^{m_n} f\left(\frac{C_{N,1}}{D_{N,1}}, \ldots, \frac{C_{N,n}}{D_{N,n}}\right) \in \mathbb{F}[x] \setminus \{0\},$$

and so

$$0 < \left| D_{N,1}^{m_1} \cdots D_{N,n}^{m_n}\ f\left(\frac{C_{N,1}}{D_{N,1}}, \ldots, \frac{C_{N,n}}{D_{N,n}}\right)\right|_\infty \leq K M_{N,1}^{m_1} \cdots M_{N,n}^{m_n},$$

where K is a positive constant depending on f but independent of N. By the product formula, we get

$$\left|f\left(\frac{C_{N,1}}{D_{N,1}},\ldots,\frac{C_{N,n}}{D_{N,n}}\right)\right|_\pi \geq \left|D_{N,1}^{m_1}\cdots D_{N,n}^{m_n}\, f\left(\frac{C_{N,1}}{D_{N,1}},\ldots,\frac{C_{N,n}}{D_{N,n}}\right)\right|_\pi$$
$$> \left|D_{N,1}^{m_1}\cdots D_{N,n}^{m_n}\, f\left(\frac{C_{N,1}}{D_{N,1}},\ldots,\frac{C_{N,n}}{D_{N,n}}\right)\right|_\infty^{-1} > \left(KM_{N,1}^{m_1}\cdots M_{N,n}^{m_n}\right)^{-1}.$$

Choosing $E = \max\{m_1, m_2, \ldots, m_n\} + 1$, by (10), there exists $N_3 = N_3(E)$ such that for all $N \geq N_3$,

$$\frac{1}{KM_{N,1}^{m_1}\cdots M_{N,n}^{m_n}} \leq \left|f\left(\frac{C_{N,1}}{D_{N,1}},\frac{C_{N,2}}{D_{N,2}},\ldots,\frac{C_{N,n}}{D_{N,n}}\right)\right|_\pi$$
$$= |\delta_n(N)H_n|_\pi \leq \frac{|H_n|_\pi}{(M_{N,1}\cdots M_{N,n})^E},$$

i.e.,

$$|H_n|_\pi \geq KM_{N,1}^{M-m_1}\cdots M_{N,n}^{M-m_n} \to \infty \qquad (N\to\infty),$$

which is a contradiction. □

Specializing the defining polynomial to be linear in each variable in the proof of Theorem 2, we get

Theorem 3. *Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}((\pi))\setminus\{0\}$. Assume that there are polynomials $C_{N,j}, D_{N,j}\ (\neq 0)\ \ (N = 1,2,3,\ldots\ ;\ 1 \leq j \leq n)$ in $\mathbb{F}[x]$ such that*

$$D_{N,j}\alpha_j \neq C_{N,j},\quad M_{N,j} := max\{|C_{N,j}|_\infty, |D_{N,j}|_\infty\} \to \infty \qquad (N\to\infty),$$

such that if $n \geq 2$,

$$\lim_{N\to\infty}\frac{|\alpha_{j-1} - C_{N,j-1}/D_{N,j-1}|_\pi}{|\alpha_j - C_{N,j}/D_{N,j}|_\pi} = 0 \qquad (j = 2,\ldots,n).$$

Assume further that there is a positive-valued function g of natural argument, with

$$g(N)\to\infty\ \ (N\to\infty),$$

and there is an $N_0 = N_0(g) \in \mathbb{N}$ such that

$$\left|\alpha_j - \frac{C_{N,j}}{D_{N,j}}\right|_\pi \leq \frac{1}{M_{N,1}M_{N,2}\cdots M_{N,j}\,g(N)} \quad (N \geq N_0\ ;\ j = 1,2,\ldots,n).$$

Then $1, \alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly independent over $\mathbb{F}(x)$.

4.1 Applications

We apply the results of Theorem 2 and 3 to derive sufficient conditions for elements in $\mathbb{F}((\pi)) \setminus \{0\}$ represented by JR-continued fractions in Example 6 above to be algebraically and linearly independent over $\mathbb{F}(x)$. Throughout we assume (without of generality) that $|\alpha|_\pi < 1$. Let

$$\frac{C_N}{D_N} = \frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\cdots\frac{b_N}{a_N} \qquad (N \in \mathbb{N})$$

be the N^{th} convergent of the JR-continued fraction expansion of α. By Proposition 1 (iv) and (v), we have

$$|C_1|_\pi = |b_1|_\pi,\ |C_i|_\pi = |b_1 a_2 a_3 \cdots a_i|_\pi \ \ (i \geq 2),\ |D_i|_\pi = |a_1 a_2 \cdots a_i|_\pi \ \ (i \geq 1),$$

which implies that $|D_i|_\pi = |a_1 C_i / b_1|_\pi > |C_i|_\pi \ \ (i \geq 1)$. Since C_N, D_N do not necessarily belong to $\mathbb{F}[x]$, to apply the results of Theorems 2 and 3, we need to convert the JR-continued fraction of Example 6 into an equivalent continued fraction.

Throughout, we focus only on the case when the sequenec $\{b_i\}$ is a subset of

$$\{\alpha = \frac{c_{-r}}{\pi^{-r}} + \cdots + \frac{c_{-1}}{\pi^{-1}} + c_o + c_1\pi^1 + \cdots + c_s\pi^s \in \mathbb{F}((\pi)) \setminus \{0\}\,;$$
$$r, s \in \mathbb{N} \cup \{0\}\,,\ c_i \in \mathbb{F}[x],\ \deg c_i < \deg \pi\}.$$

For each $i \in \mathbb{N}$, write

$$a_i := \frac{a_i^{'}}{\pi^{n_i}},\ \ b_i := \frac{b_i^{'}}{\pi^{m_i}},$$

where $n_i \in \mathbb{N} \cup \{0\}, m_i \in \mathbb{Z}$, and $a_i^{'}, b_i^{'} \in \mathbb{F}[x]$ are both relatively prime to π, so that

$$|a_i^{'}|_\pi = 1 = |b_i^{'}|_\pi,\ |a_i|_\pi = e^{n_i \deg \pi},\ |b_i|_\pi = e^{m_i \deg \pi}.$$

From the ab-condition, we have $n_i > m_i \ \ (i \in \mathbb{N})$. It is convenient to introduce an associated JR-continued fraction

$$\frac{\gamma_1}{\beta_1+}\,\frac{\gamma_2}{\beta_2+}\cdots\frac{\gamma_i}{\beta_i}\cdots, \tag{11}$$

where

$$\gamma_1 = b_1^{'}\pi^{n_1 - m_1},\ \gamma_{i+1} = b_{i+1}^{'}\pi^{n_i + n_{i+1} - m_{i+1}},\ \ \beta_i = a_i^{'} \qquad (i \in \mathbb{N}).$$

Clearly, the partial numerators γ_i and the partial denominators β_i of the associated continued fraction (11) are in $\mathbb{F}[x]$ and $|\gamma_i|_\infty > e^{\deg \pi}$ $(i \geq 1)$. We similarly define the N^{th} convergent of (11) to be

$$\frac{\mathcal{C}_N}{\mathcal{D}_N} = \frac{\gamma_1}{\beta_1+}\,\frac{\gamma_2}{\beta_2+}\cdots\frac{\gamma_N}{\beta_N} \qquad (N \in \mathbb{N}),$$

where

$$\mathcal{C}_{-1} = 1, \quad \mathcal{C}_0 = 0, \quad \mathcal{C}_{i+1} = \beta_{i+1}\mathcal{C}_i + \gamma_{i+1}\mathcal{C}_{i-1} \quad (i \geq 0)$$

$$\mathcal{D}_{-1} = 0, \quad \mathcal{D}_0 = 1, \quad \mathcal{D}_{i+1} = \beta_{i+1}\mathcal{D}_i + \gamma_{i+1}\mathcal{D}_{i-1} \quad (i \geq 0).$$

The JR-continued fraction (8) and its associated continued fraction (11) are equivalent in the sense that $C_N/D_N = \mathcal{C}_N/\mathcal{D}_N$ ($N \in \mathbb{N}$). Clearly, $\mathcal{C}_N$ and $\mathcal{D}_N$ are in $\mathbb{F}[x]$. In what follows, we assume that

$$|\beta_i\beta_{i+1}|_\infty > |\gamma_{i+1}|_\infty, \qquad (i \in \mathbb{N}), \tag{12}$$

which is equivalent to $|a_i a_{i+1}|_\infty > |b_{i+1}|_\infty$, where $|\cdot|_\infty$ denote the degree valuation mentioned in Example 5. The next lemma summarizes basic properties of $\mathcal{C}_N$ and $\mathcal{D}_N$, whose induction proof is omitted.

Lemma 1. *Let the notation be as above. If* (12) *holds, then*

(i) $|\mathcal{C}_1|_\infty = |\gamma_1|_\infty, \quad |\mathcal{C}_N|_\infty = |\mathcal{C}_1\beta_2\cdots\beta_N|_\infty \quad (N \geq 2)$

(ii) $|\mathcal{D}_N|_\infty = |\beta_1\beta_2\cdots\beta_N|_\infty \qquad (n \in \mathbb{N})$

(iii) $M_N := \max\{|\mathcal{C}_N|_\infty, |\mathcal{D}_N|_\infty\} \to \infty \quad (N \to \infty)$.

Let $\alpha_j \in \pi\mathbb{F}[[\pi]] \setminus \{0\}$ $\;(1 \leq j \leq k)$ with associated JR-continued fractions

$$\alpha_j = \frac{\gamma_{1,j}}{\beta_{1,j}+}\,\frac{\gamma_{2,j}}{\beta_{2,j}+}\cdots,$$

and let their corresponding N^{th} convergent be

$$\frac{\mathcal{C}_{N,j}}{\mathcal{D}_{N,j}} = \frac{\gamma_{1,j}}{\beta_{1,j}+}\,\frac{\gamma_{2,j}}{\beta_{2,j}+}\cdots\,\frac{\gamma_{N,j}}{\beta_{N,j}} \qquad (1 \leq j \leq k,\ N \in \mathbb{N}).$$

If the requirement (12) holds for each $j \in \{1, \ldots, k\}$, then Lemma 1 yields

$$M_{N,j} = \max\{|\mathcal{C}_{N,j}|_\infty, |\mathcal{D}_{N,j}|_\infty\} \to \infty \qquad (N \to \infty).$$

Theorem 4. *Let the notation be as bove. Assume that*

I. the condition (12) *is fulfilled for each* $j \in \{1, \ldots, k\}$;

II. the limiting values

$$\lim_{i\to\infty} \frac{|b_{1,j-1}b_{2,j-1}\cdots b_{i+1,j-1}|_\pi |D_{i,j}D_{i+1,j}|_\pi}{|b_{1,j}b_{2,j}\cdots b_{i+1,j}|_\pi |D_{i,j-1}D_{i+1,j-1}|_\pi} = 0 \qquad (2 \leq j \leq k) \tag{13}$$

hold and

III. there exists $g : \mathbb{N} \to \mathbb{Z}$ *with* $g(i) \to \infty$ $(i \to \infty)$ *such that*

$$\frac{|D_{i,j} D_{i+1,j}|_\pi}{|b_{1,j} b_{2,j} \cdots b_{i+1,j}|_\pi} \geq (M_{i,1} M_{i,2} \cdots M_{i,j})^{g(i)} \quad (1 \leq j \leq k;\ i \in \mathbb{N}). \quad (14)$$

Then $\alpha_1, \alpha_2, \ldots, \alpha_k$ *are algebraically independent over* $\mathbb{F}(x)$.

Moreover, if the condition (14) *is replaced by*

$$\frac{|D_{i,j} D_{i+1,j}|_\pi}{|b_{1,j} b_{2,j} \cdots b_{i+1,j}|_\pi} \geq g(i) \left(M_{i,1} M_{i,2} \cdots M_{i,j} \right),$$

then $1, \alpha_1, \alpha_2, \ldots, \alpha_k$ *are linearly independent over* $\mathbb{F}(x)$.

Proof. For a fixed $E > 0$, from $g(i) \to \infty$ $(i \to \infty)$, there is $N_0 \in \mathbb{N}$ such that for all $N > N_0$, we have $g(N) > E$. For $1 \leq j \leq k$ and $N > N_0$, applying (14), we get

$$\begin{aligned} \left| \alpha - \frac{\mathcal{C}_{N,j}}{\mathcal{D}_{N,j}} \right|_\pi &= \left| \alpha - \frac{C_{N,j}}{D_{N,j}} \right|_\pi = \frac{|b_{1,j} b_{2,j} \cdots b_{N+1,j}|_\pi}{|D_{N,j} D_{N+1,j}|_\pi} \\ &\leq \frac{1}{(M_{N,1} M_{N,2} \cdots M_{N,j})^{g(N)}} < \frac{1}{(M_{N,1} M_{N,2} \cdots M_{N,j})^{E}}. \end{aligned} \quad (15)$$

From (13), we get

$$\begin{aligned} \frac{|\alpha_{j-1} - \mathcal{C}_{N,j-1}/\mathcal{D}_{N,j-1}|_\pi}{|\alpha_j - \mathcal{C}_{N,j}/\mathcal{D}_{N,j}|_\pi} &= \frac{|\alpha_{j-1} - C_{N,j-1}/D_{N,j-1}|_\pi}{|\alpha_j - C_{N,j}/D_{N,j}|_\pi} \\ &= \frac{|b_{1,j-1} b_{2,j-1} \cdots b_{N+1,j-1}|_\pi / |D_{N,j-1} D_{N+1,j-1}|_\pi}{|b_{1,j} b_{2,j} \cdots b_{N+1,j}|_\pi / |D_{N,j} D_{N+1,j}|_\pi} \to 0 \quad (N \to \infty). \end{aligned} \quad (16)$$

Noting (15) and (16), Theorem 2 yields the desired result of the first part. The second part follows using similar arguments but appealing instead to Theorem 3. □

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